Subformula Replacement

In the exercises, we have seen the following:

Theorem:

Let F_1, F_2 are formulas. Let A be an n-place predicate symbol not occurring in F_1 or F_2. Then F_1 has a model iff $F_1 \wedge \forall x_1 \cdots x_n \, [A(x_1, \ldots, x_n) \leftrightarrow F_2]$ has a model.

Take for F_2 a subformula of F_1 and take for $x_1, \ldots, x_n$ all variables that have an occurrence free in F_2 but bound in F_1.

One can write F_1 in the form $F_1[F_2]$ (to stress that F_2 occurs in F_1)

Theorem: $F_1[F_2]$ has a model iff
$F_1[A(x_1, \ldots, x_n)],\ \forall x_1 \cdots x_n\ A(x_1, \ldots, x_n) \leftrightarrow F_2$ has a model.

$F_1[A(x_1, \ldots, x_n)]$ means: The result of replacing F_1 by $A(x_1, \ldots, x_n)$. (on the position that we had in mind when we wrote $F_1[F_2]$)

How do you prove this? You probably guess 'by induction', but what is the induction hypothesis?

Let $x_1, \ldots, x_n$ be a set of variables containing at least the variables that are free in F_2 but bound in F_1. Then, etc.

Theorem:

If F_2 is not in the scope of $\neg$, not in the scope of $\leftrightarrow$, not inside a left argument of $\rightarrow$, then

$F_1[F_2]$ has a model iff
$F_1[A(x_1, \ldots, x_n)]$, $\forall x_1 \cdots x_n \; [\; A(x_1, \ldots, x_n) \rightarrow F_2 \;]$ has a model.

Applications:

- Exponential increase with NNF-transformation can be avoided.
- It is possible to take a formula completely apart.

Skolemization

Let F be a formula in NNF. Assume that F contains an existential quantifier.

Write F in the form $F[\ \exists y\ A\ ]$, where $\exists y\ A$ is not in the scope of another existential quantifier.

Let $x_1, \ldots, x_n$ be the free variables of $\exists y\ A$ that have an occurrence that is free in $\exists y\ A$, but bound in F.

Let f be a new function symbol with arity n. Then $F[\ A[y := f(x_1, \ldots, x_n)]\ ]$ is called a Skolemization of $F[\ \exists y A\ ]$, and f is called a Skolem function.

Theorem Let F be a formula, F' be a Skolemization. Then F has a model iff F' has a model.

For simplicity, one can first apply subformula replacement on $\exists y\ A$.

The result is
$F[\ B(x_1, \ldots, x_n)\ ],\ \forall x_1 \cdots x_n\ [\ B(x_1, \ldots, x_n) \rightarrow \exists y\ A\ ]$.

The intuition for $\Rightarrow$ is:

If, for some $d_1, \ldots, d_n,\ (d_1, \ldots, d_n) \notin [B]$, then $f(d_1, \ldots, d_n)$ does not matter. Otherwise, there is an $e \in D$, s.t. $(d_1, \ldots, d_n, e) \in [A]$. Choose such an e, and set $[f](d_1, \ldots, d_n) = e$.

For $\Leftarrow$, observe that the sequent

$$\forall x_1 \cdots x_n \; [\; B(x_1, \ldots, x_n) \to A[y := f(x_1, \ldots, x_n)] \;]$$

$$\vdash$$

$$\forall x_1 \cdots x_n \; [\; B(x_1, \ldots, x_n) \to \exists y \; A]$$

is provable.

For $\Rightarrow$, (now comes a real proof) suppose that

$$F,\ \forall x_1 \cdots x_n [\ B(x_1, \ldots, x_n) \rightarrow \exists y\ A\ ]$$

has a model, that f does not occur in F or A. We show that this implies that
$F,\ \forall x_1 \cdots x_n [\ B(x_1, \ldots, x_n) \rightarrow \exists y\ (\ y = f(x_1, \ldots, x_n) \wedge A\ )\ ]$ has a model.

Run **ModSearch**$(\{c_0\},\ \{F,\ \forall x_1 \cdots x_n\ [\ B(x_1, \ldots, x_n) \rightarrow \exists y\ A]\ \}\ )$.

Because the formula has a model, ModSearch will construct or approximate a pair $(\mathcal{C}^*, \Gamma^*)$ with the properties mentioned in the completeness proof.

Define the following interpretation $(D, [\ ])$:

- $D = \mathcal{C}$.
- For each $f \in \mathcal{F} \backslash \mathcal{C}$ with arity n, for $c_1, \ldots, c_n \in \mathcal{C}$, there is exactly one equality of form $f(c_1, \ldots, c_n) = d$ in Γ^*. Put $[f](c_1, \ldots, c_n) = d$.
- For each $p \in \mathcal{P}$ with arity n, for $c_1, \ldots, c_n \in \mathcal{C}$, put

$$(c_1, \ldots, c_n) \in [p] \text{ iff } p(c_1, \ldots, c_n) \in \Gamma.$$

(until here, everything is the same as in the completeness proof)

Let f be the Skolem function.

For each n-tuple $(c_1, \ldots, c_n) \in \mathcal{C}^{*n}$,

- If $(\exists A)[x_1 := c_1, \ldots, x_n := c_n] \notin \Gamma^*$, then

$$[f](c_1, \ldots, c_n) = c,$$

  where c is the first element in $\mathcal{C}$.

- If $(\exists A)[x_1 := c_1, \ldots, x_n := c_n] \in \Gamma^*$, then let $\hat{c}$ be the first element in $\mathcal{C}$, for which $A[x_1 := c_1, \ldots, x_n := c_n, y := \hat{c}] \in \Gamma^*$. (It follows from the properties of Γ^* mentioned in the completeness proof, that at least one such $\hat{c}$ exists)

At this point, it is not so hard to check that

1. $(D, [\,]) \models F$,
2. $(D, [\,]) \models$

$$\forall x_1 \cdots x_n [\ B(x_1, \ldots, x_n) \rightarrow \exists y\ (\ y = f(x_1, \ldots, x_n) \wedge A\ )\ ].$$

It remains to prove the sequent

$$\forall x_1 \cdots x_n [\ B(x_1, \ldots, x_n) \rightarrow \exists y\ (\ y = f(x_1, \ldots, x_n) \wedge A\ )\ ]$$

$$\vdash$$

$$\forall x_1 \cdots x_n [\ B(x_1, \ldots, x_n) \rightarrow A[y := f(x_1, \ldots, x_n)]\ ].$$

The Pain of Choosing (Die Qual der Wahl)

Why did we not stay in the original interpretation, and choose $[f]$ immediately there?

Because there is no way of choosing in an arbitrary domain D without using the axiom of choice:

Let F be a set of non-empty sets. ($\forall S\ S \in F \rightarrow \exists s\ s \in S$)

There exists a set C consisting of pairs (S, s) with $s \in S$, s.t. for each $S \in F$ there is an s, s.t. $(S, s) \in F$.

The problem arises when the elements of $\bigcup F$ do not have enough structure, so that they cannot be distinguished by a formula. In that case, the usual set forming principle (comprehension) cannot be used to define a set $\mathcal{C}$.

By referring to the completeness proof, we can choose within the set $\mathcal{C}^*$, which has enough structure to make the choice possible, because the elements have been introduced in a well-defined order.