Semantics for Propositional Logic

An interpretation (also truth-assignment, valuation) of a set of propositional formulas S is a function that assigns elements of $\{\mathbf{f}, \mathbf{t}\}$ to the propositional variables in S.

The function can be partial, but it must assign values to the propositional variables in S.

Truth of a Formula in Propositional Logic

The following table defines how the propositional logical operators propagate their truth values:

A	B	$\bot$	$\top$	$\neg A$	$A \vee B$	$A \wedge B$	$A \rightarrow B$	$A \leftrightarrow B$
f	**f**	**f**	**t**	**t**	**f**	**f**	**t**	**t**
f	**t**	**f**	**t**	**t**	**t**	**f**	**t**	**f**
t	**f**	**f**	**t**	**f**	**t**	**f**	**f**	**f**
t	**t**	**f**	**t**	**f**	**t**	**t**	**t**	**t**

Truth of a Formula in an Interpretation

Consider the truth assignment I_1, defined by

$$I_1(P) = \mathbf{f}, \quad I_1(Q) = \mathbf{t}, \quad I_1(R) = \mathbf{t},$$

and I_2, which is defined by

$$I_2(P) = \mathbf{t}, \quad I_2(Q) = \mathbf{f}, \quad I_2(R) = \mathbf{t}.$$

Then:

$$I_1(P \land (Q \rightarrow R)\ ) = \mathbf{f}, \quad I_2(P \land (Q \rightarrow R)\ ) = \mathbf{t},$$

$$I_1(Q \leftrightarrow R) = \mathbf{t}, \quad I_2(Q \leftrightarrow R) = \mathbf{f},$$

$$I_1(\neg P \rightarrow \neg Q) = \mathbf{f}, \quad I_2(\neg P \rightarrow \neg Q) = \mathbf{t}.$$

Some Basic Definitions:

Definition: Let F be a formula, let I be an interpretation. If $I(F) = \mathbf{t}$, then I is a model of F. (also called: satisfying truth-assignment, satisfying interpretation) One usually writes $I \models F$.

Definition: A formula is universally valid if for every interpration I,

$$I \models F$$

A formula G is a consequence of $F_1, \ldots, F_n$, if for every interpretation I, for which

$$I \models F_1, \ldots, I \models F_n, \text{ also } I \models G.$$

One usually writes $F_1, \ldots, F_n \models G$.

Examples

The following formulas are universally valid:

$$\neg A \vee A, \quad A \rightarrow A, \quad (\neg\neg A) \leftrightarrow A, \quad (A \vee B) \wedge (\neg A \vee C) \rightarrow B \vee C.$$

The right hand sides are consequences of the left hand sides:

$$\neg\neg A \models A,$$

$$A \rightarrow B,\ B \rightarrow C \models \neg C \rightarrow \neg A,$$

$$\neg(A \vee B) \models \neg A \wedge \neg B,$$

$$\neg(A \wedge B) \models \neg A \vee \neg B,$$

$$\neg(A \rightarrow B) \models A \wedge \neg B,$$

$$A, \neg A \models B.$$

Correctness of Sequent Calculus

Definition: A sequent $\Gamma \vdash \Delta$ is valid if

- for every interpretation I, s.t. for all $A \in \Gamma, \quad I(A) = \mathbf{t}$,
- there is a $B \in \Delta$, s.t. $I(B) = \mathbf{t}$.

Theorem: Every provable sequent is valid.

proof: It is enough to show preservence of validity for each of the rules. We do a few on the next slides:

Correctness of $\neg$-left

Assume that $\Gamma \vdash \Delta, A$ is a valid sequent We show that $\Gamma, \neg A \vdash \Delta$ is a valid sequent as well.

In order to do this, we assume an arbitrary truth-assignment I, which makes all formulas $F \in \Gamma \cup [\neg A]$ true, and show that it makes one of the formulas $G \in \Delta$ true.

Assume that I is a truth-assignment, s.t. for all $F \in \Gamma, \quad I(F) = \mathbf{t}$, and $I(\neg A) = \mathbf{t}$. From the truth-table of $\neg$, it follows that $I(A) = \mathbf{f}$.

Because $\Gamma \vdash \Delta, A$ is a valid sequent, there must be a formula $G \in \Delta \cup [A]$, s.t. $I(G) = \mathbf{t}$. Since $I(A) = \mathbf{f}$, it cannot be A. Therefore it must be a $G \in \Delta$.

Correctness of $\vee$-left

Assume that $\Gamma, A \vdash \Delta$ and $\Gamma, B \vdash \Delta$ are valid sequents. We show that $\Gamma, A \vee B \vdash \Delta$ is a valid sequent as well.

Let I be an interpretation, s.t. for all $F \in \Gamma \cup [A \vee B]$, we have $I(F) = \mathbf{t}$. In particular, $I(A \vee B) = \mathbf{t}$. From the truth-table of $\vee$, we see that either $I(A) = \mathbf{t}$, or $I(B) = \mathbf{t}$.

If $I(A) = \mathbf{t}$, then all $F \in \Gamma \cup [A]$ are true. Therefore, we can use the validity of the first sequent to obtain that one $G \in \Delta$ is true.

Similarly, if $I(B) = \mathbf{t}$, all $F \in \Gamma \cup [B]$ are true, and we can use the validity of the second sequent.

In both cases, there is a $G \in \Delta$, s.t. $I(G) = \mathbf{t}$.

Completeness of Sequent Calculus

For every deduction system, (at least when its logic has a well-defined semantics), one can ask the following questions:

correctness/soundness Is every provable object valid?

completeness Is every valid object provable?

In general, correctness is much more important than completeness. Many logics/deduction systems that are used in practice are not complete.

Completeness of Sequent Calculus

We define a algorithm Chck($\Gamma \vdash \Delta$) that either returns a proof of $\Gamma \vdash \Delta$, or a counter interpretation.

- If $\Gamma \vdash \Delta$ contains only propositional variables, then there are two possiblities:
  1. $\Gamma \cap \Delta \neq \emptyset$. $\Gamma \vdash \Delta$ can be written in form $\Gamma', A \vdash \Delta', A$. Return the proof that consists of the single axiom
  $$\Gamma', A \vdash \Delta', A.$$
  2. $\Gamma \cap \Delta = \emptyset$. Construct a truth-assignment I as follows: For all $F \in \Gamma,\ I(F) = \mathbf{t}$, for all $G \in \Delta,\ I(G) = \mathbf{f}$. Return I.

- If $\Gamma \vdash \Delta$ has form $\Gamma', \bot \vdash \Delta$, then return the proof consisting of the single axiom $\Gamma`, \bot \vdash \Delta$.
- If $\Gamma \vdash \Delta$ has form $\Gamma \vdash \Delta', \bot$, then let

$$\pi = \mathrm{Chck}(\Gamma \vdash \Delta').$$

If π is an interpretation, then return π. Otherwise, return

$$\frac{\pi}{\Gamma \vdash \Delta', \bot}(\text{weakening right})$$

- If $\Gamma \vdash \Delta$ has form $\Gamma \vdash \Delta', \neg A$, then let

  $$\pi = \mathrm{Chck}(\Gamma, A \vdash \Delta').$$

  If π is an interpretation, then return π. Otherwise, return

  $$\frac{\pi}{\Gamma \vdash \Delta', \neg A}(\neg\text{-right})$$

- If $\Gamma \vdash \Delta$ has form $\Gamma', \neg A \vdash \Delta$, then let

  $$\pi = \mathrm{Chck}(\Gamma' \vdash \Delta, A).$$

  If π is an interpretation, then return π. Otherwise, return

  $$\frac{\pi}{\Gamma', \neg A \vdash \Delta}(\neg\text{-left})$$

- If $\Gamma \vdash \Delta$ has form $\Gamma \vdash \Delta',\ A \vee B$, then let

$$\pi = \mathrm{Chck}(\Gamma \vdash \Delta', A, B).$$

If π is an interpretation, then return π. Otherwise, return

$$\frac{\pi}{\Gamma \vdash \Delta', A \vee B}(\vee\text{-right})$$

- If $\Gamma \vdash \Delta$ has form $\Gamma', A \vee B \vdash \Delta$, then let

$$\pi_1 = \mathrm{Chck}(\Gamma', A \vdash \Delta), \quad \pi_2 = \mathrm{Chck}(\Gamma', B \vdash \Delta).$$

If π_1 is an interpretation, then return π_1. If π_2 is an interpretation, then return π_2.

(Both of them are proofs) Return

$$\frac{\pi_1 \qquad\qquad \pi_2}{\Gamma', A \vee B \vdash \Delta}(\vee\text{-left})$$

Excercise

Fill in the missing cases.

Properties of Algorithm Chck

- Algorithm Chck terminates.
- If Chck($\Gamma \vdash \Delta$) returns an interpretation I, then I is a counter interpretation of $\Gamma \vdash \Delta$.
- If Chck($\Gamma \vdash \Delta$) returns a proof π, then π is a proof of $\Gamma \vdash \Delta$.

Completeness of sequent calculus follows from the correctness of Chck.

Some More Observations/Questions:

- In case, more than one rule can be applied, non-branching rules should be preferred over branching rules.
- Not all rules of propositional sequent calculus are used by algorithm Chck. Which rules? What can be concluded?
- What is the time complexity of Chck?
- What is its space complexity?