Introduction to Formal Logic: Exercises about Inductive Sets

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1. We define the sets of even and odd numbers:

$$\Phi_E := \lambda P{:}\,\mathrm{Nat} \rightarrow \mathrm{Prop}\ (P\ 0) \wedge (\ \forall n{:}\,\mathrm{Nat}\ (P\ n) \rightarrow (P\ (\mathrm{succ}\ (\mathrm{succ}\ n)))\ ).$$

$$E := \lambda n{:}\,\mathrm{Nat}\ \forall P{:}\,\mathrm{Nat} \rightarrow \mathrm{Prop}\ (\Phi_E\ P) \rightarrow (P\ n).$$

$$\Phi_O := \lambda P{:}\,\mathrm{Nat} \rightarrow \mathrm{Prop}\,(P\ (\mathrm{succ}\ 0)) \wedge (\ \forall n{:}\,\mathrm{Nat}\ (P\ n) \rightarrow (P\ (\mathrm{succ}\ (\mathrm{succ}\ n)))\ ).$$

$$O := \lambda n{:}\,\mathrm{Nat}\ \forall P{:}\,\mathrm{Nat} \rightarrow \mathrm{Prop}\ (\Phi_O\ P) \rightarrow (P\ n).$$

   (a) Determine the type of Φ_E.
   (b) Determine the type of E.
   (c) Prove $(E\ (\mathrm{succ}\ (\mathrm{succ}\ 0)))$ in natural deduction.
   (d) Prove that $\neg(E\ (\mathrm{succ}\ 0))$. (This is non-trivial. You will need the axioms that Nat is freely generated)
   (e) Prove that $\forall n{:}\,\mathrm{Nat}\ (E\ n) \wedge (O\ n) \rightarrow \bot$. (Note: This task is very hard)

2. The order $\leq$ on natural numbers can be defined as follows:

   $\Phi_{\leq} := \lambda n{:}\,\mathrm{Nat}\ \lambda P{:}\,\mathrm{Nat} \rightarrow \mathrm{Prop}\ \ (P\ n) \wedge \forall m{:}\,\mathrm{Nat}\ (P\ m) \rightarrow (P\ (\mathrm{succ}\ m)).$

   Then $\leq := \lambda m, n{:}\,\mathrm{Nat}\ \forall P{:}\,\mathrm{Nat} \rightarrow \mathrm{Prop}\ (\Phi_{\leq}\ m\ P) \rightarrow (P\ n).$

   (a) Show that $\leq$ is reflexive.
   (b) Show that $\leq$ is transitive.
   (c) Show that $\forall m, n{:}\,\mathrm{Nat}\ (\leq\ \ m\ n) \rightarrow (\leq\ \ n\ m) \rightarrow m = n$. (And this again is a challenge, requiring the fact that Nat is freely generated)