Exercises

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10.01.2008

1. On the slides about Higher-Order logic, two alternative definitions of $=$ are given.

   First consider the first definition:

   $$= \;:=\; \lambda D{:}\,\mathrm{Type}\ \lambda d_1, d_2{:}\,D$$
   $$\forall P{:}\,D \to \mathrm{Prop}\ (P\ d_1) \to (P\ d_2){:}\,\Pi D{:}\,\mathrm{Type}\ D \to D \to \mathrm{Prop}$$

   (a) Show, using the type derivation rules for HOL, that the defining term of $=$ has the indicated type.

   (b) Prove in natural deduction that $=$ is reflexive:
   $$\forall D{:}\,\mathrm{Type} \forall d{:}\,D\ (=\ \ D\ d\ d).$$

   (c) Prove in natural deduction that $=$ is symmetric:
   $$\forall d{:}\,D\ \forall d_1, d_2{:}\,D\ (=\ \ D\ d_1\ d_2) \to (=\ \ D\ d_2\ d_1).$$

   (The definition of $=$ is the replacement rule. In usual natural deduction, symmetry for equality can be derived using the replacement rule. Transform this proof into a higher-order logic proof)

2. $$=_2 \;:=\; \lambda D{:}\,\mathrm{Type} \lambda d_1, d_2{:}\,D$$
   $$\forall P{:}\,D \to D \to \mathrm{Prop}\ (\ \forall d{:}\,D\ (P\ d\ d)\ ) \to (P\ d_1\ d_2){:}$$
   $$\Pi D{:}\,\mathrm{Type} D \to D \to \mathrm{Prop}$$

   Write $=_1$ for the first definition of equality. Write $=_2$ for the second.

   (a) Prove that
   $$\forall D{:}\,\mathrm{Type}\ \forall d_1, d_2{:}\,D\ (=_1\ \ D\ d_1\ d_2) \to (=_2\ \ D\ d_2\ d_1).$$

   (b)
   $$\forall D{:}\,\mathrm{Type}\ \forall d_2, d_1{:}\,D\ (=_2\ \ D\ d_2\ d_1) \to (=_1\ \ D\ d_1\ d_2).$$