Exercises

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1. Prove the following sequents in natural deduction:

   (a) $A \vee B,\ \neg A \vee \neg C \vdash B \vee \neg C$,

   (b) $(A \rightarrow Q) \rightarrow (B \rightarrow Q) \rightarrow Q,\quad (A \rightarrow C \rightarrow Q) \vdash (B \rightarrow Q) \rightarrow C \rightarrow Q$,

   (c) $(\forall x\ P(x)\ ) \wedge \exists y\ Q(y) \vdash \exists y\ [P(y) \wedge Q(y)]$,

   (d) $(\forall x\ P(x)\ ) \wedge \exists y\ Q(y) \vdash \forall x \exists y\ [\ P(x) \wedge Q(y)\ ]$,

   (e) $\vdash (\ \exists x\ P(x)\ ) \leftrightarrow (\ \exists x\ P(x) \wedge Q(x)) \vee (\ \exists x\ P(x) \wedge \neg Q(x))$,

   (f) $\forall xy\ P(x,y) \vee Q(x,y),\ \ \forall xy\ \neg P(x, s(x)) \vee R(x) \vdash \forall x\ Q(x, s(x)) \vee R(x)$,

   (g) $\exists x\ B(x),\ \forall xy\ A(x) \wedge B(y) \wedge x \neq y \rightarrow \bot\ \vdash\ \forall x\ A(x) \rightarrow B(x)$,

   (h) $\exists x\ (\ D(x) \rightarrow \forall y\ D(y)\ )$. (the Drinker visits us one more time)

   (i) $P(0), \neg P(s(s(0))) \vdash \exists x\ P(x) \wedge \neg P(s(x))$.

   (It is certain that the last two formulas have no proof. (Why not?) It is possible that (g) has no proof, and (e) only in one direction, we will see)

2. Full classical logic can be recovered in case one of the following two reasonaing principles is added to natural deduction:

   - Tertium non datur (no third is given/law of excluded middle)

     $A \vee \neg A$

   - Double negation elimination:

     $\neg(\neg A)$

     A

   Show that the law of excluded middle is provable from double negation elimination, and the reverse. (The proofs are tricky, in the next lecture we will see the general pattern)

3. Using TND or DNE, complete the missing proofs of Exercise 1. (The real lesson is that natural deduction and classical logic are not created for each other)