Introduction to Formal Logic: Exercises on Semantics of Predicate Logic

25 October 2007

1. Consider the interpretation $(D, [\ ])$ defined by $D = \mathcal{N}$, and

$$[0] = 0,$$

$$[s] \text{ is defined by } [s](n) = n + 1.$$

   $[f]$ is defined by

$$[f](n) = n/2 \text{ if } n \text{ is even }, \quad [f](n) = 3n + 1 \text{ if } n \text{ is odd }.$$

   Give the interpretations of the following terms in I.

   (a) $[f(s(0))]$,
   (b) $[s(s(s(0)))]$,
   (c) Does $I \models s(0) = f(f(f(s(0))))$?
   (d) Does $I \models \exists y\ f(f(f(y))) = y$?
   (e) Does $I \models \exists y\ f(y) = s(y)$?
   (f) Does $I \models \exists y\ f(y) = y$?

2. Consider the interpretation $I = (D, [\ ])$ defined by

$$D = \{0, 1, 2, 3, 4, 5, 6\},$$

$$[R] = \{(0, 1), (2, 3), (3, 4), (5, 6)\},$$

$$[S] = \{(0, 2), (1, 3), (3, 5), (4, 6)\}.$$

   (Hint: Make a drawing of the interpretation)

   (a) Is the formula $\exists xy\ R(x, y)$ true in I?
   (b) And $\forall x \exists y\ R(x, y)$?
   (c) $\forall xy\ R(x, y) \to \exists z\ S(y, z)$?
   (d) $\forall xyz\ R(x, y) \wedge S(x, z) \to \exists t\ S(y, t) \wedge R(z, t)$?

(e) $\forall xyz R(x,y) \wedge R(y,z) \rightarrow S(x,y) \wedge S(y,x)$?

3. (a) Find an interpretation in which the formula $\forall x \exists y\ R(x,y)$ is true.

   (b) Find an interpretation in which all of the following formulas are true: $\forall x\ \neg R(x,x)$, $\forall xyz\ R(x,y) \wedge R(y,z) \rightarrow R(x,z)$, $\forall x\ \exists y\ R(x,y)$.

   (c) Consider the following sequent

$$P(0),\ \forall x\ P(x) \rightarrow P(s(x)) \vdash \forall x\ P(x).$$

   Give a counter interpretation of this sequent.

   (d) And of the following sequent:

$$P(0),\ \forall x\ P(x) \rightarrow P(s(x)), \forall x\ (x = 0 \vee \exists y\ s(y) = x) \vdash \forall x\ P(x).$$

   (We will see later that natural numbers are not definable in first-order logic)

4. Prove the correctness of the $\exists$-right rule of sequent calculus.

5. (a) Consider the following interpretations I_1 and I_2 defined by

$$D_1 = \{0,1,2\},\ [E]_1 = \{(0,1),\ (1,2),(2,0)\}$$

   and

$$D_2 = \{0,1,2,3\},\ [E]_2 = \{(0,1),(1,2),(2,3),(3,0)\}.$$

   Find a closed formula F that is true in D_1 but not in D_2.