Introduction to Formal Logic: Exercises on Sequent Calculus

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1. (a) $f(g(x), x)[x := g(y)]$
   (b) $f(g(x), y)[x := g(y)]$
   (c) $p(f(y, x), g(x))[y := x]$
   (d) $(\ \forall y\ p(f(y, x), g(y))\ )\ [y := x]$
   (e) $(\ \forall y\ p(f(y, x), x)\ )[x := y]$
   (f) $(\ \forall y\ (\ p(f(z, g(y)), g(x))\ ) \lor (\exists z\ g(z) = y)\ )\ [x := g(y)]$,
   (g) $\forall y\ f(x, y) = x \to \forall x\ (\ f(x, y) = x\ )\ [y := g(y)][x := g(z)]$
2. Write the formulas d,e,f,g of Task 1 in de Bruijn notation. Carry out the substitutions, using de Bruijn notation.
3. Let D be a one place predicate. Express the following properties:
   (a) There are at least three Ds.
   (b) There are at most three Ds.
4. Prove the following sequents:
   (a) $\forall x\ x = x$,
   (b) $\forall x\ s(x) = x \vdash \forall x\ s(s(s(x))) = x$.
   (c) $\forall x\ P \to Q(x) \vdash P \to \forall x\ Q(x)$.
   (d) $\forall x\ (P(x) \land Q\ ) \vdash (\forall x\ P(x)\ ) \land Q$,
   (e) $(\neg\forall x\ P(x)) \leftrightarrow \exists x\ \neg P(x)$,
   (f) $\exists x\ (\ D(x) \to \forall y\ D(y)\ )$, (the Drinker paradox)
   (g) $\forall x\ f(x) = x,\quad \forall x \exists y\ p(f(x), y) \vdash \forall x \exists y\ p(x, f(y))$.
5. Prove the following sequents:
   (a) Let

$$\text{DEFLEQ} = \forall xy\ x \leq y \leftrightarrow [\ P(x) \land (\ \forall n\ P(n) \to P(s(n))\ ) \to P(y)\ ]$$

Prove the sequents

$$\mathrm{DEFLEQ} \vdash x \leq x,$$

$$\mathrm{DEFLEQ}, x \leq y, y \leq z \vdash x \leq z.$$

(b) Let

$$\mathrm{PLUSDEF1} = \forall x\ x + 0 = x,$$

$$\mathrm{PLUSDEF2} = \forall xy\ x + s(y) = s(x + y).$$

Prove the sequents

$$\mathrm{PLUSDEF1},\ \mathrm{PLUSDEF2} \vdash 0 + 0 = 0,$$

$$\mathrm{PLUSDEF1},\ \mathrm{PLUSDEF2} \vdash \forall y\ 0 + y = y \rightarrow 0 + s(y) = s(y).$$

(c) Prove the sequents

$$\mathrm{PLUSDEF1},\ \mathrm{PLUSDEF2} \vdash s(0) + 0 = s(0),$$

$$\mathrm{PLUSDEF1},\ \mathrm{PLUSDEF2} \vdash \forall xy\ s(x){+}y = s(x{+}y) \rightarrow s(x){+}s(y) = s(x{+}s(y)).$$