Natural Deduction and Intuitionistic Logic

There are two important families of deduction systems. The two most important are:

- Natural Deduction: Natural deduction follows the natural style of reasoning in text books. Most of an ND proof consists of forward reasoning. Occassionally additional assumptions are introduced or dropped. Natural deduction treats conclusions and assumptions different. Because of this, it fits better to intuitionistic logic than to classical logic.

- Sequent Calculus: In sequent calculus, conclusions and premisses are treated in exactly the same way. Because of this, sequent calculus naturally fits to classical logic. Automated proof search in sequent calculus is relatively easy. Each time when a rule is applied, most formulas are copied unchanged, and one has to write very much.

Intuitionistic Logic the logic that one obtains (from classical logic) when one drops the laws of excluded middle ($A \vee \neg A$) and the law of double negation ($\neg\neg A \rightarrow A$)

Intuitionistic logic has the witness property. That means: If one proves a formula of form $\forall x\ A(x) \vee B(x)$, then one can determine for every t, which of $A(t) \vee B(t)$ holds. Similarly, if one has proven a formula of form $\forall x\ \exists y\ P(x, y)$, one can extract from the proof, for every t, a term u for which $P(t, u)$ holds.

As a consequence, in intuitionistic logic, proofs of formulas of form $\forall x\ \exists y\ p(x, y)$ define algorithms.

One also says: Intuitionistic logic is constructive, because everything that one speaks about can be constructed.

Classical logic does not have the witness property:

$$p(0), \neg p(s(s(s(0)))) \vdash \exists x \; (\; p(x) \wedge \neg p(s(x)) \;).$$

The cause is contraction. Using Herbrand's theorem, the proof has to be constructed through

$$p(0), \neg p(s(s(s(0)))) \vdash$$

$$\exists x \; p(x) \wedge \neg p(s(x)), \quad \exists x \; p(x) \wedge \neg p(s(x)), \quad \exists x \; p(x) \wedge \neg p(s(x)).$$

This gives three possible witnesses $s(0), \; s(s(0)), \;\; s(s(s(0)))$.

Natural Deduction proofs have form

$$\begin{array}{|l} \text{Assumptions} \\ \hline \text{Conclusions.} \end{array}$$

The assumptions have to be formulas. The conclusions are also formulas. It is possible to introduce more assumptions in the conclusion-part of the proof. In this case, one obtains a nested proof, as for example in

$$\begin{array}{|l} \text{Assumptions} \\ \hline \text{Conclusions.} \\ \begin{array}{|l} \text{More assumptions} \\ \hline \text{More conclusions.} \end{array} \\ \text{More assumptions} \rightarrow \text{More conclusions.} \end{array}$$

The conclusions are derived from the assumptions using forward reasoning rules. For example if one has derived A and one has derived B, then one may conclude $A \wedge B$ and add this formula to the conclusions.

Natural Deduction for Intuitionistic Logic

Rules for the constants:

$\top$-introduction:

$$\begin{array}{|l} \ldots \\ \top \end{array}$$

$\perp$-elimination:

$$\begin{array}{|l} \perp \\ \ldots \\ A \end{array}$$

Rules for $\neg$:

$\neg$-introduction:

$$
\begin{array}{|l}
\begin{array}{|l}
A \\ \hline
\dots \\
\bot
\end{array} \\
\neg A
\end{array}
$$

$\neg$-elimination:

$$
\begin{array}{|l}
A \\
\dots \\
\neg A \\
\dots \\
\bot
\end{array}
$$

Rules for $\wedge$:

$\wedge$-elimination:

$$\begin{array}{|l}A \wedge B \\ \ldots \\ A\end{array} \qquad \begin{array}{|l}A \wedge B \\ \ldots \\ B\end{array}$$

$\wedge$-introduction:

$$\begin{array}{|l}A \\ \ldots \\ B \\ \ldots \\ A \wedge B\end{array}$$

Rules for $\vee$:

$\vee$-elimination: (reasoning by cases)

$$
\begin{array}{|l}
A \vee B \\
\ldots \\
\begin{array}{|l}
A \\ \hline
\ldots \\
C
\end{array} \\
\ldots \\
\begin{array}{|l}
B \\ \hline
\ldots \\
C
\end{array} \\
C
\end{array}
$$

Rules for $\vee$ (continued):

$\vee$-introduction(left and right):

A $\ldots$ $A \vee B$	B $\ldots$ $A \vee B$

Rules for $\leftrightarrow$:

$\leftrightarrow$-elimination:

$A \leftrightarrow B$	$A \leftrightarrow B$
$\ldots$	$\ldots$
$A \rightarrow B$	$B \rightarrow A$

$\leftrightarrow$-introduction:

$A \rightarrow B$

$\ldots$

$B \rightarrow A$

$\ldots$

$A \leftrightarrow B$

Rules for $\rightarrow$:

$\rightarrow$-elimination:

A

$\ldots$

$A \rightarrow B$

$\ldots$

B

$\rightarrow$-introduction:

A

$\ldots$

B

$A \rightarrow B$

Rules for $\forall$:

$\forall$-elimination:

$$\forall x\ P(x)$$

$$\ldots$$

$$P[x := t]$$

$\forall$-introduction:

x is a variable that is not free in an assumption.

$$\ldots$$

$$P(x)$$

$$\forall x\ P(x)$$

Rules for $\exists$:

$\exists$-elimination:

$$
\begin{array}{|l}
\exists x \; P(x) \\
\ldots \\
\begin{array}{|l}
x \text{ is a variable that is not free in assumption.} \\
P(x) \\
\hline
\ldots \\
C
\end{array} \\
C
\end{array}
$$

(C must not contain x)

Rules for $\exists$ (continued): $\exists$-introduction:

$P[x := t]$

$\ldots$

$\exists x\ P(x)$

Rules for $=$:

Equality reflexivity:

$$\begin{array}{|l} \ldots \\ t = t \end{array}$$

Equality replacement:

$$\begin{array}{|l} t_1 = t_2 \\ \ldots \\ P(t_1) \\ \ldots \\ P(t_2) \end{array}$$

(or in the other direction)

Remarks:

Small variations in the rules are possible. For example $\vee$-elim, $\exists$-elim, $\neg$-rules.

Natural deduction naturally corresponds to intuitionistic logic. Sequent calculus naturally corresponds to classical logic, but it can be adapted to intuitionistic logic.