Sequent Calculus for Intuitionistic Logic,

Equivalence of Sequent Calculus and Natural Deduction for intuitionistic logic

Sequent Calculus for Intuitionistic Logic:

We have seen (from examples) that the non-constructivity of classical logic is due to contraction on the right.

A radical solution is the following:

An intuitionistic sequent is an object of form

$$\Gamma \vdash \Delta, \ \text{with } |\Delta| \leq 1.$$

So, the multiset Δ is either empty, or it contains a single formula.

An intuitionistic sequent has form $\Gamma \vdash \Delta$, where Δ is either empty, or contains one formula. In case Δ is empty, we can put $\Delta = \bot$, so we assume that Δ always consists of exactly one formula.

From now on an intuitionistic sequent will have form $\Gamma \vdash B$, where Γ is a multiset of formulas, and B a single formula.

The rules for intuitionistic sequent calculus are (almost) the same as the rules for sequent calculus for classical logic.

Rules for intuitionistic sequent calculus:

Propositional Rules:

$$(\text{axiom})\ \frac{}{\Gamma, A \vdash A}$$

$$(\text{cut})\ \frac{\Gamma \vdash A \qquad \Gamma, A \vdash B}{\Gamma \vdash B}$$

Structural Rules:

$$\text{(weakening left)}\ \frac{\Gamma \vdash B}{\Gamma, A \vdash B} \qquad \text{(weakening right)}\ \frac{\Gamma \vdash \bot}{\Gamma \vdash B}$$

$$\text{(contraction left)}\ \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}$$

(There is no contraction on the right:

$$\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

because the premisse is not a valid sequent)

Axioms for the constants:

$$(\top\text{-right})\ \frac{}{\Gamma \vdash \top}$$

$$(\bot\text{-left})\ \frac{}{\Gamma, \bot \vdash B}$$

Rules for $\neg$:

$$(\neg\text{-left }) \ \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash \bot} \qquad\qquad (\neg\text{-right }) \ \frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}$$

Rules for $\wedge$:

$$(\wedge\text{-left }) \ \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \qquad\qquad (\wedge\text{-right }) \ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

Rules for $\vee$:

$$(\vee\text{-left})\ \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}$$

$$(\vee\text{-right1})\ \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad\qquad (\vee\text{-right2})\ \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

The rules $\vee$-right12 are the additive rules for $\vee$-right for classical logic. The usual multiplicative rule $\Gamma \vdash A, B \ \Rightarrow\ \Gamma \vdash A \vee B$ has too many conclusions on the right in the premisse.

Rules for $\longrightarrow$:

$$(\longrightarrow\text{-left}) \; \frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \longrightarrow B \vdash C}$$

$$(\longrightarrow\text{-right}) \; \frac{\Gamma, A \vdash B}{\Gamma \vdash A \longrightarrow B}$$

Rules for $\leftrightarrow$::

$$(\leftrightarrow\text{-left}\ )\ \frac{\Gamma, A \rightarrow B,\ \ B \rightarrow A \vdash C}{\Gamma, A \leftrightarrow B \vdash C}$$

$$(\leftrightarrow\text{-right}\ )\ \frac{\Gamma \vdash A \rightarrow B \qquad \Gamma \vdash B \rightarrow A}{\Gamma \vdash A \leftrightarrow B}$$

Rules for the quantifiers:

$$(\forall\text{-left }) \ \frac{\Gamma, P[x := t] \vdash B}{\Gamma, \ \forall x \ P(x) \vdash B} \qquad\qquad (\forall\text{-right }) \ \frac{\Gamma \vdash P}{\Gamma \vdash \forall x \ P}$$

$$(\exists\text{-left }) \ \frac{\Gamma, \ P \vdash B}{\Gamma, \exists x \ P \vdash B} \qquad\qquad (\exists\text{-right }) \ \frac{\Gamma \vdash P[x := t]}{\Gamma \vdash \exists x \ P}$$

t is an arbitrary term. The rules $\forall$-right and $\exists$-left have the eigenvariable condition: x must be not free in Γ or B.

rules for equality:

$$(\mathrm{refl})\ \frac{}{\Gamma \vdash t = t}$$

$$(\mathrm{repl})\ \frac{t_1 = t_2,\ \Gamma[t_1] \vdash B[t_1]}{t_1 = t_2,\ \Gamma[t_2] \vdash B[t_2]}$$

The last rule means: If $t_1 = t_2$ appears among the premisses, then arbitrary free occurrences of t_2 in other formulas can be replaced by t_1.

(Replacing t_2 by t_1 is also allowed)

Some Anatomy of Natural Deduction Proofs

An object of form

is called a scope. The part above the line is the assumption part. It consists of formulas and possible declarations of fresh variables.

The part below the line is the derivation part. The derivation part consists of formulas and other scopes.

If a formula F is assumed in the assumption part of some scope S, then all formulas in the derivation part of S and in the derivation parts of scopes inside S are derived under the assumption F.

Examples

$$
\begin{array}{|l}
A_1 \\
A_2 \\ \hline
A_3 \\
\begin{array}{|l} A_4 \\ \hline A_5 \end{array} \\
A_6 \\
\begin{array}{|l} A_7 \\ \hline A_8 \end{array} \\
A_9
\end{array}
$$

A_6 is derived under the assumptions A_1, A_2. Formula A_8 is derived under the assumptions A_1, A_2, A_7.

A More Practical Notation for ND-Proofs

Since ND-proofs are rather space consuming, we introduce a new notation for ND-proofs. We write a scope

$$\begin{array}{|l} A_1 \\ A_2 \\ \hline F_1 \\ F_2 \end{array}$$

as $[\ A_1, A_2 \mid F_1, F_2\ ]$.

Theorem: Intuitionistic Sequent Calculus and Natural Deduction have the same strength. More precisely:

1. If $\Gamma \vdash B$ has a proof in intuitionistic sequent calculus, then one can construct a natural deduction proof of form $[\ \Gamma \mid \pi_1, \ldots, \pi_n, B\ ]$.
2. For every natural deduction proof in which a formula B is derived under assumptions $A_1, \ldots, A_n$, the sequent $A_1, \ldots, A_n \vdash B$ has a proof in intuitionistic sequent calculus.

We first prove (1) by induction on the structure of the proof:

We first do the structural rules:

Permutation:

If $[\ \Gamma \mid \pi_1, \ldots, \pi_n, B\ ]$ is a natural deduction proof, and Γ' is a permutation of Γ, then $[\ \Gamma' \mid \pi_1, \ldots, \pi_n, B\ ]$ is also a natural deduction proof.

(Without this property, it would even be problematic to define a function from intuitionistic sequents to natural deduction proofs)

Contraction:

If $[\ \Gamma, A, A \mid \pi_1, \ldots, \pi_n, B\ ]$ is a natural deduction proof, then $[\ \Gamma, A \mid \pi_1, \ldots, \pi_n, B\ ]$ is also a natural deduction proof.

Weakening:

If there is a natural deduction proof of form

$$[\ \Gamma \mid \pi_1, \ldots, \pi_n, B\ ]$$

then there exists a natural deduction proof of form

$$[\ \Gamma, A \mid \pi_1', \ldots, \pi_n', B\ ].$$

This is almost trivial, but there is the usual problem with fresh variables: It may happen that a fresh variable, which was used in an $\exists$-elim or $\forall$-intro, is free in A.

The solution is the usual solution: Replace the fresh variable in the proof by another variable. This can be viewed as defining α-equivalence on natural deduction proofs.

Cut:

Natural deduction has built-in cut, because it cannot distinguish assumptions from formulas that have been proven before.

Let

$$[\ \Gamma \mid \pi_1, \ldots, \pi_m, A\ ],$$

and

$$[\ \Gamma, A \mid \rho_1, \ldots, \rho_n, B\ ]$$

be natural deduction proofs.

Then

$$[\ \Gamma \mid \pi_1, \ldots, \pi_m, A, \rho_1, \ldots, \rho_n, B\ ]$$

is a natural deduction proof.

Axioms:

Consider the axiom $\Gamma, A \vdash A$. One can construct the natural deduction proof $[\ \Gamma, A \mid A\ ]$.

Next consider the axiom $\Gamma, \bot \vdash B$. Using $\bot$-elimination, one can construct the natural deduction proof $[\ \Gamma, \bot \mid B\ ]$.

Finally, consider axiom $\Gamma \vdash \top$. Using $\top$-introduction, one can construct the natural deduction proof $[\ \Gamma \mid \top\ ]$.

Consider the $\wedge$-left rule $\Gamma, A, B \vdash C \quad \Rightarrow \quad \Gamma, A \wedge B \vdash C$.
By the induction hypothesis, there is a natural deduction proof

$$[\Gamma, A, B \mid \pi_1, \ldots, \pi_n, C].$$

One can also construct the proofs

$$[\ A \wedge B \mid A\ ]\ ]\text{ and }[\ A \wedge B \mid B\ ]\ ].$$

Using two times cut, one time contraction, and some weakening (in ND), one can construct the proof

$$[\Gamma, A \wedge B \mid A, B, \pi_1, \ldots, \pi_n, C].$$

Consider the $\wedge$-right rule $\Gamma \vdash A, \quad \Gamma \vdash B \quad \Rightarrow \quad \Gamma \vdash A \wedge B$.
By induction, there are natural deduction proofs

$$[\ \Gamma \mid \pi_1, \ldots, \pi_n, A\ ] \text{ and } [\ \Gamma \mid \rho_1, \ldots, \rho_m, B\ ].$$

One can construct the following natural deduction proof

$$\Gamma \mid \pi_1, \ldots, \pi_n, A, \rho_1, \ldots, \rho_m, B, A \wedge B].$$

Consider the $\vee$-left rule $\Gamma, A \vdash C, \quad \Gamma, B \vdash C \quad \Rightarrow \quad \Gamma, A \vee B \vdash C$. By induction, there are natural deduction proofs

$$[\ \Gamma, A \mid \pi_1, \ldots, \pi_n, C\ ], \text{ and}$$

$$[\ \Gamma, B \mid \rho_1, \ldots, \rho_m, C\ ].$$

From this one can construct the following proof:

$$[\ \Gamma, A \vee B \mid [\ A \mid \pi_1, \ldots, \pi_n, C\ ],\ [\ B \mid \rho_1, \ldots, \rho_m, C\ ], C\ ].$$

The other propositional rules follow the same pattern, and we skip them.

Consider the $\forall$-left rule $\Gamma,\ P[x := t] \vdash A \quad \Rightarrow \quad \Gamma, \forall x\ P \vdash A$. By induction, one can assume that there is a natural deduction proof of form

$$[\ \Gamma, P[x := t] \mid \pi_1, \ldots, \pi_n, A\ ],$$

One can construct $[\ \forall x\ P \mid P[x := t]\ ]$, and use cut to obtain

$$[\ \Gamma, \forall x\ P \mid P[x := t], \pi_1, \ldots, \pi_n, A\ ].$$

Now consider the $\forall$-right rule $\Gamma \vdash P \quad \Rightarrow \quad \Gamma \vdash \forall x\ P$. By induction, one can construct a proof of form $[\ \Gamma \mid \pi_1, \ldots, \pi_n, P\ ]$. Because x is not free in Γ, one can construct the natural deduction proof

$$[\ \Gamma \mid \pi_1, \ldots, \pi_n, P,\ \forall x\ P\ ].$$

Consider the $\exists$-left rule $\Gamma, P \vdash A \quad \Rightarrow \quad \Gamma, \exists x\ P \vdash A$. By induction, one can assume that there is a natural deduction proof of form

$$[\ \Gamma, P \mid \pi_1, \ldots, \pi_n, A\ ].$$

From this, one can construct the proof

$$[\ \Gamma, \exists x\ P \mid [P \mid \pi_1, \ldots, \pi_n, A\ ], A\ ],$$

using the fact that x is not free in Γ or A.

Now consider the $\exists$-right rule $\Gamma \vdash P[x := t] \quad \Rightarrow \quad \Gamma \vdash \exists x\ P$. By induction, one can construct the proof $[\ \Gamma \mid \pi_1, \ldots, \pi_n, P[x := t]\ ]$. One can also construct the proof $[\ P[x := t] \mid \exists x\ P\ ]$, and use cut to obtain the natural deduction proof

$$[\ \Gamma \mid \pi_1, \ldots, \pi_n,\ P[x := t],\ \exists x\ P\ ].$$

Equality replacement

Finally consider the rule

$t_1 = t_2,\ \Gamma[t_1] \vdash B[t_1] \quad \Rightarrow \quad t_1 = t_2,\ \Gamma[t_2] \vdash B[t_2]$.

By induction hypothesis, we assume that there is a natural deduction proof of form $[\ t_1 = t_2,\ \Gamma[t_1] \mid \pi_1, \ldots, \pi_n,\ B[t_1]\ ]$.

Using equality replacement in ND, one can construct proofs

$$[\ t_1 = t_2,\ B[t_1] \mid B[t_2]\ ] \text{ and } [\ t_1 = t_2,\ \Gamma[t_2] \mid \rho_1, \ldots, \rho_m,\ \Gamma[t_1]\ ].$$

Using cut, one can obtain a proof of the form

$$[\ t_1 = t_2,\ \Gamma[t_2] \mid \rho_1, \ldots, \rho_m,\ \Gamma[t_1],\ \pi_1, \ldots, \pi_n,\ B[t_2]\ ].$$

We now prove (2) as follows: Fix a natural deduction proof Π. Then prove for every formula A that occurs in Π and which is not an assumption, the following fact: If for every formula that occurs before A and which is not an assumption, (2) already holds, then (2) also holds for A.

This fact is proven by a case analysis on the rule that was used to derive A.

Assume that A is obtained using $\wedge$-elim. Then there is a formula $A \wedge B$ before A in the proof, which has the same set of assumptions as A. Call the set of assumptions Γ. By induction, one can obtain a sequent proof for $\Gamma \vdash A \wedge B$. Since the sequent $A \wedge B \vdash A$ is provable, one can use the cut1-rule to prove $\Gamma \vdash A$.

Now assume that $A \wedge B$ is obtained by $\wedge$-intro. Then there are formulas A and B before $A \wedge B$ in the proof, having the same set of assumptions Γ. By induction, one may assume that there are proofs of $\Gamma \vdash A$ and $\Gamma \vdash B$. On these, one can apply $\wedge$-right to obtain a proof of $\Gamma \vdash A \wedge B$.

There is a general pattern: Introduction rules become right introduction rules. Elimination rules need to be proven using the cut1-rule.

Assume that C is obtained by $\vee$-elim from a formula $A \vee B$. The formulas $A \vee B$ and C have the same set of assumptions, call them Γ. So, we have a proof of form

$$[\,\cdots, [\,\ldots, A \vee B, [\; A \mid \;\cdots, C\;],\; [\; B \mid \;\cdots, C\;], C\;] \cdots].$$

By induction, one can obtain sequent proofs for
$\Gamma \vdash A \vee B, \quad \Gamma, A \vdash C$, and $\Gamma, B \vdash C$. Applying $\vee$-left on the last two sequents, one obtains $\Gamma, A \vee B \vdash C$. Applying cut1, one obtains $\Gamma \vdash C$.

Now assume that a formula of form $A \vee B$ is obtained using $\vee$-intro. There must be an earlier formula A which has the same set of assumptions Γ. By induction, one may assume that there is a proof of $\Gamma \vdash A$. One can apply $\vee$-right to obtain a proof of $\Gamma \vdash A \vee B$.

(The case where $A \vee B$ is derived from B goes analogeously)

Assume that the rule is $\rightarrow$-elim. We have a formual of form B that obtained from $A \rightarrow B$ and A.

By induction, one can obtain sequent proofs for $\Gamma \vdash A \rightarrow B$ and $\Gamma \vdash A$. The sequent $A, A \rightarrow B \vdash B$ is provable. Applying cut1 two times, one obtains a proof of $\Gamma \vdash B$.

Now assume that the rule is $\rightarrow$-intro. The relevant part of the proof has form

$$[\cdots [\ A \mid\ \cdots, B\ ],\ A \rightarrow B, \cdots].$$

Assume that the context of $A \rightarrow B$ equals Γ. Then the context for B equals Γ, A. By induction, one may assume that there is a proof of $\Gamma, A \vdash B$. One can apply $\rightarrow$-right to obtain a proof of $\Gamma \vdash A \rightarrow B$.

Assume that the rule is $\forall$-elim. Then $P[x := t]$ is obtained from $\forall x\ P$ and both formulas have the same set of assumptions Γ.

By induction, there exists a sequent proof for $\Gamma \vdash \forall x\ P$. The sequent $\forall x\ P \vdash P[x := t]$ is provable. Applying cut1, one obtains a proof of $\Gamma \vdash P[x := t]$

Now assume that the last rule is $\forall$-intro. Then $\forall x\ P$ is derived from P, which has the same context as $\forall x\ P$, but an additional eigenvariable condition for x. By induction, the sequent $\Gamma \vdash P$ is provable. Because of the eigenvariable condition, one can apply $\forall$-right, and obtain a proof of $\Gamma \vdash \forall x\ P$.

Assume that C is obtained by $\exists$-elim. There is a formula $\exists x\ P$ under the same assumptions as C, which occurs earlier in the proof. Call the set of assumptions Γ.

$$\ldots, \exists x\ P,\ [\ P \mid \cdots, C\ ],\ C\ ], \cdots$$

By induction, one can obtain proofs for the sequents $\Gamma \vdash \exists x\ P$ and $\Gamma, P \vdash C$. Because of the eigenvariable condition of $\exists$-elim, one can apply $\exists$-left on the last sequent, and obtain $\Gamma, \exists x\ P \vdash C$. Using cut1, one obtains a proof of $\Gamma \vdash C$.

Assume that the rule is $\exists$-intro. We have a formula of form $\exists x\ P$ that is obtained from an earlier formula $P[x := t]$ which has the same set of assumptions Γ. By induction, one may assume that there is a proof of $\Gamma \vdash P[x := t]$. One can apply $\exists$-right on this proof and obtain a proof of $\Gamma \vdash \exists x\ P(x)$.

This shows that intuitionistic sequent calculus and natural deduction have equal strength.

Intuitionistic logic is strictly weaker (but actually richer) than classical logic. For example, the formulas $A \vee \neg A$ and $(\neg\neg A \rightarrow A)$ are not provable.

This can be shown by introducing a semantics, (and prove completeness), or by showing cut elimination for intuitionistic sequent calculus.

Regaining Classical Logic

Classical Logic can be obtained back by adding one of the following two axiom schemas to intuitionistic sequent calculus:

For each formula A, $\qquad \vdash \neg\neg A \rightarrow A.$

For each formula A, $\qquad \vdash \neg A \vee A.$

The first rule is called double negation elimation (DNE). The second rule is called law of excluded middle, or tertium non datur (TND)

It is easy to see that using these axioms, one cannot prove more than in classical logic, because intuitionistic sequent calculus was obtained by restricting classical sequent calculus, and the two axioms are provable in classical sequent calculus.

We now show the reverse: Every formula that is provable in classical logic, is provable in intuitionistic logic + DNE or intuitionistic logic + TND.

We show that the following 3 statements are equivalent:

1. Sequent $\Gamma \vdash \Delta_1, \ldots, \Delta_n$ is is provable in classical sequent calculus.
2. Sequent $\Gamma, \neg\Delta_1, \ldots, \neg\Delta_n \vdash \bot$ is provable in intuitionistic sequent calculus with TND or DNE.
3. (In case $n > 0$) Sequent $\Gamma, \neg\Delta_1, \ldots, \neg\Delta_{n-1} \vdash \Delta_n$ is provable in intuitionistic sequent calculus with TND or DNE.

We first show that (2) $\Leftrightarrow$ (3). Note that, using this equivalence, every $\neg\Delta_i$ can be moved to the right, because one can always permute the Δ_i.

The implication (3) $\Rightarrow$ (2) is a simple application of $\neg$-left.

In order to show (2) $\Rightarrow$ (3), we first show that from either of the axioms, it is possible to derive the sequent $\neg\neg A \vdash A$, for every formula A.

For DNE, first prove $\neg\neg A \to A,\ \neg\neg A \vdash A$ in intuitionistic sequent calculus. After that use cut with $\vdash \neg\neg A \to A$.

For TND, first prove the sequent

$$A \vee \neg A,\ \neg\neg A \vdash A$$

in intuitionistic sequent calculus.

Then use cut with $\vdash A \vee \neg\neg A$.

Now assume that $\Gamma, \neg\Delta_1, \ldots, \neg\Delta_n \vdash \bot$ has a proof. Use $\neg$-right to obtain a proof of $\Gamma, \neg\Delta_1, \ldots, \neg\Delta_{n-1} \vdash \neg\neg\Delta_n$.

Then use cut with $\neg\neg\Delta_n \vdash \Delta_n$ to obtain $\Gamma, \neg\Delta_1, \ldots, \Delta_{n-1} \vdash \Delta_n$.

(2) $\Rightarrow$ (1) can be proven as follows. From (2) follows that $\Gamma, \neg\Delta_1, \ldots, \neg\Delta_n$ also has a proof in classical sequent calculus.

From $\Gamma, \neg\Delta_1, \ldots, \neg\Delta_n \vdash \bot$ prove $\Gamma \vdash \neg\neg\Delta_1, \ldots, \neg\neg\Delta_n$ in classical logic.

Use $\neg\neg\Delta_i \vdash \Delta_i$ to remove the negations.

(1) $\Rightarrow$ (2) can be shown by observing the following:

All left rules of classical sequent calculus are valid rules of intuitionistic sequent calculus. Most of the right rules of classical sequent calculus are also valid in intuitionistic sequent calculus in case Δ is empty.

In case Δ is not empty, one can move the remaining formulas in Δ to the left, using $\neg$-left. If later one want to apply a right introduction rule on a formula that has been moved to the left, one can get it back using (2) $\Leftrightarrow$ (3).

There are only two problematic rules, which work on two formulas at the same time to the right. These are contraction right, and $\vee$-right.

Contraction right

$$\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\text{(contr-right)}$$

can be replaced by contraction left:

$$\frac{\Gamma, \neg\Delta, \neg A, \neg A \vdash \bot}{\Gamma, \neg\Delta, \neg A \vdash \bot}\text{(contr-left)}$$

$\vee$-right

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}(\vee\text{-right}$$

can be replaced by

$$\frac{\Gamma, \neg A \vdash B}{\Gamma \vdash A \vee B},$$

which is provable.

Alternatively, in classical sequent calculus, one can replace the multiplicative rule by the two additive rules:

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}(\vee\text{-right1}) \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}(\vee\text{-right2})$$

The multiplicative $\vee$-right is provable from $\vee$-right1 and $\vee$-right2, by using contraction.

If one moves Δ to the left, $\vee$-right12 are the same as the $\vee$-right rules of intuitionistic sequent calculus.

Summary

- Natural deduction defines the same logic as sequent calculus in case only one formula is allowed on the right. The logic is called intuitionistic logic.
- By adding either tertium non datur, or double negation elimination to intuitionistic logic, classical logic can be obtained back.