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The reconstruction of polyominoes from their orthogonal projections

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Abstract

The reconstruction of discrete two-dimensional pictures from their projections is one of the central problems in the areas of medical diagnostics, computer-aided tomography, pattern recognition, image processing and data compression. In this note, we determine the computational complexity of two open problems in this field; one of our results settles a long-standing open question. We will prove that it is NP-complete to reconstruct a two-dimensional pattern from its two orthogonal projections H and V, if (1) the pattern has to be connected (and hence forms a so-called *polyomino*), or if (2) the pattern has to be horizontally and vertically convex. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

1.1. Problem statement

A *finite binary picture* is an $m \times n$ matrix of 0's and 1's. Intuitively speaking, the 1's correspond to black pixels (which constitute the *pattern*) and the 0's correspond to white pixels (which form the *background*). The ith *row projection* and the jth *column projection* are the numbers of black pixels in the ith row and in the jth column, respectively. In a *pattern reconstruction problem*, we are given two vectors $H = (h_1, \dots, h_m) \in \mathbb{N}^m$ and $V = (v_1, \dots, v_n) \in \mathbb{N}^n$, and we want to decide whether there exists a picture in which the ith row projection equals h_i and in which the jth column projection equals v_j. Often, the reconstructed pattern should fulfill several additional properties like symmetry, connectivity, or convexity. In this note, we will mainly deal with the three properties of being *horizontally convex* (in every row the black pixels form a contiguous interval), *vertically convex* (in every column the black pixels form an interval), and *connected* (connected in the usual sense: every pixel is adjacent to its two vertical and to its two horizontal neighbors, and the set of black pixels has to be connected with respect to this adjacency relation). A connected pattern is called a *polyomino* (cf. Golomb [7]).

1.2. Known results

Ryser [9], and subsequently Chang [2], studied the pattern reconstruction problem without extra restric-

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Table 1
The computational complexity of pattern reconstruction

	Horizontal/vertical convexity			
	$v+h$ convex	v convex	h convex	no restriction
Polyomino	P[b,c]	NP-complete[b]	NP-complete[b]	NP-complete[d]
No restriction	NP-complete[d]	NP-complete[b]	NP-complete[b]	P[a]

The result with superscript [a] was derived by Ryser. Results with superscript [b] were derived by Barcucci, Del Lungo, Nivat and Pinzani. The result [c] is by Chrobak and Dürr. The results [d] are derived in the current paper.

tions imposed onto the combinatorial structure of the pattern. They gave an exact combinatorial characterization of the projections (H, V) that correspond to a pattern, and they derived a fast $O(mn)$ time algorithm that outputs a picture that is compatible with (H, V). Chang and Chow [3] defined an algorithm that reconstructs patterns that are convex and symmetrical with respect to the two orthogonal axes. Del Lungo [5] investigated a special case where the pattern fulfills a certain "north-east connectivity" constraint.

Kuba [8] designed a heuristic algorithm for reconstructing patterns that are horizontally and vertically convex. Barcucci, Del Lungo, Nivat and Pinzani [1] performed a careful investigation on the computational complexity of different variants of the pattern reconstruction problem. As a main result, they derived a polynomial time algorithm for the reconstruction of horizontally and vertically convex polyominoes. Moreover, they proved that the four reconstruction problems for (1) horizontally convex polyominoes, (2) vertically convex polyominoes, (3) horizontally convex patterns, and (4) vertically convex patterns all are NP-complete. Chrobak and Dürr [4] presented a faster and simpler polynomial time algorithm for the reconstruction of horizontally and vertically convex polyominoes.

1.3. Our results

In this note, we settle the two questions that were left open by Barcucci et al. [1]: First, we will prove that the reconstruction of polyominoes is an NP-complete problem. Then we show that a slight modification of our argument also yields NP-completeness of the reconstruction of vertically and horizontally convex patterns; this answers an open problem that dates back to the paper [8] by Kuba. See Table 1 for a summary of all known complexity results.

2. Reconstruction of polyominoes

Our first reduction is done from the following version of the NP-complete THREE PARTITION problem (Garey and Johnson [6]).

Problem. THREE PARTITION

Instance. Positive integers $a_1, \ldots, a_{3k}$ that are encoded in unary and that fulfill the two conditions

(i) $\sum_{i=1}^{3k} a_i = k(2B+1)$ for some integer B, and
(ii) $(2B+1)/4 < a_i < (2B+1)/2$ for $1 \leqslant i \leqslant 3k$.

Question. Does there exist a partition of $a_1, \ldots, a_{3k}$ into k triples such that the elements of every triple add up to exactly $2B+1$?

Now let an instance of THREE PARTITION be given. From this instance, we will construct a row vector $H^* \in \mathbb{N}^m$ and a column vector $V^* \in \mathbb{N}^n$ with $m = (2B+2)k+1$ and $n = 12k$ such that the following holds: There exists a polyomino with row projections H^* and column projections V^*, if and only if the THREE PARTITION instance has a solution. Clearly, this will establish NP-completeness of the reconstruction problem for polyominoes.

The vector H^* starts with a single entry $h_1^* = n$, followed by k identical blocks, each block consisting of $2B+2$ numbers. Every block starts with the entries $3k$ and $3k+2$, then there come $B-1$ entries equal to $3k+1$, then a single entry $3k+2$ followed by another sequence of $B-1$ entries equal to $3k+1$, and then the block ends with another single entry $3k+2$. Hence, it is of the form

$$(n, \dots, 3k, 3k+2, \\ \underbrace{3k+1, \dots, 3k+1}_{(B-1)\text{-times}}, 3k+2, \\ \underbrace{3k+1, \dots, 3k+1}_{(B-1)\text{-times}}, 3k+2, \dots).$$

The vector V^* consists of $3k$ blocks each with four entries, where the ith block equals $1, a_i+1, 2, m$, i.e., V^* is of the form

$$(1, a_1+1, 2, m,\ 1, a_2+1, 2, m, \\ 1, a_3+1, 2, m,\ 1, a_4+1, 2, m, \dots, \\ 1, a_{3k}+1, 2, m).$$

We will say that for $1 \leqslant i \leqslant 3k$, the columns $4i-3$, $4i-2$, $4i-1$ and $4i$ constitute the ith *column block* (or *c-block*) C_i, and that for $1 \leqslant j \leqslant k$, the rows from $(2B+2)(j-1)+2$ to $(2B+2)j+1$ form the jth *row block* (or *r-block*) R_j.

Lemma 1. *If the* THREE PARTITION *instance has a solution, then there exists a polyomino with row projections H^* and column projections V^*.*

Proof. We start with coloring black all the pixels in row 1 and all the pixels in the columns $4i$, $1 \leqslant i \leqslant 3k$. Next, let $a_{j(1)}$, $a_{j(2)}$, and $a_{j(3)}$ be the elements in the jth triple in the solution of the THREE PARTITION instance. In column $4j(1)-2$, we introduce a black interval of length $a_{j(1)}$ that extends from row $(2B+2)(j-1)+3$ to row $(2B+2)(j-1)+a_{j(1)}+2$; in column $4j(2)-2$, we put a black interval of length $a_{j(2)}$ that extends from row $(2B+2)(j-1)+a_{j(1)}+3$ to row $(2B+2)(j-1)+a_{j(1)}+a_{j(2)}+2$; in column $4j(3)-2$, we put a black interval of length $a_{j(3)}$ that extends from row $(2B+2)(j-1)+a_{j(1)}+a_{j(2)}+2$ to row $(2B+2)j+1$. Moreover, we blacken the three pixels at the intersection of column $4j(1)-1$ with row $(2B+2)(j-1)+3$, at the intersection of column $4j(2)-1$ with row $(2B+2)(j-1)+B+3$, and at the intersection of column $4j(3)-1$ with row $(2B+2)(j-1)+2B+3$. This procedure is repeated for every triple in the solution of the THREE PARTITION instance. It is easy to verify that the resulting pattern is connected and has H^* and V^* as row and column projections, respectively. □

Lemma 2. *If there exists a polyomino with row projections H^* and column projections V^*, then the* THREE PARTITION *instance has a solution.*

Proof. Consider the polyomino with row projections H^* and column projections V^*: In every c-block C_i, $1 \leqslant i \leqslant 3k$, the rightmost column must be completely black, as the corresponding projection $v^*_{4i} = m$ equals the number of rows. This fixes the positions of $3k$ black pixels in every row. Since every r-block R_j starts with a row with exactly $3k$ black pixels, we know that the black pixels in the rows $(2B+2)(j-1)+2$, $1 \leqslant j \leqslant k$, lie exactly at the intersections with the columns $4i$, $1 \leqslant i \leqslant 3k$. Similarly, the first row must be completely black, and this fixes the positions of the single black pixel in every column $4i-3$, $1 \leqslant i \leqslant 3k$, and the position of one of the two black pixels in every column $4i-1$.

Now let us take a closer look at the c-block C_i (consisting of columns $4i-3$, $4i-2$, $4i-1$ and $4i$). We claim that the a_i+1 black pixels in column $4i-2$ form two disjoint intervals: one single pixel in row 1, and a contiguous interval of a_i black pixels that is completely contained within *one* r-block. Because of the discussion in the preceding paragraph, we already know about the black pixel in row 1, and we know that there is no black pixel at the crossing with row 2. Now suppose that there are at least two further black intervals in this column $4i-2$. Both of these intervals must somehow be connected to the rest of the polyomino. This connection cannot happen via the left neighboring column (column $4i-3$ only has a black pixel at the crossing with row 1) and hence, it must happen via the right neighboring column $4i-1$. The right neighboring column, however, contains only two black pixels, one in row 1, and another one that can only connect a single interval in column $4i-2$ to the rest of the polyomino. Hence, column $4i-2$ indeed contains an interval of a_i black pixels. Finally, this single interval cannot be part of two r-blocks, say R_j and R_{j+1}: In this case it would have to cross the starting row $(2B+2)j+2$ of r-block R_{j+1}. By the above discussion, we know that there is no black pixel at the intersection of this row with column $4i-2$. This completes the proof of the claim.

Next, let us define a partition $\mathcal{P}^*$ of the numbers a_i into k parts: Number a_i belongs to part j if and only if the interval of length a_i in column $4i-2$

belongs to the r-block R_j. Block R_j contains exactly $(3k+1)(2B+2)+2$ black pixels. From the above claim we get that the intersection of R_j with c-Block C_i either contains exactly $3k$ black pixels (in column $4i$) or it does contain exactly $3k+a_i+1$ black pixels ($3k$ in column $4i$, a_i in column $4i-2$, and a single connecting pixel in column $4i-1$). With this it is easy to see that if part j in the partition $\mathcal{P}^*$ contains x elements, then these elements sum up to $2B+4-x$. By property (ii) of the THREE PARTITION instance, $x=3$ must hold for every part, and thus the partition $\mathcal{P}^*$ constitutes a solution to the THREE PARTITION instance. □

Putting together the statements in Lemmas 1 and 2 yields our main result.

Theorem 3. *The reconstruction of polyominoes from their orthogonal projections is* NP-*complete.*

3. Reconstruction of vertically and horizontally convex patterns

Our second reduction is done from the following version of the NP-complete problem NUMERICAL MATCHING WITH TARGET SUMS (Garey and Johnson [6]).

Problem. NUMERICAL MATCHING WITH TARGET SUMS

Instance. Positive integers $a_1,\dots,a_k,b_1,\dots,b_k$, and $c_1,\dots,c_k$ that are encoded in unary. For some integer D and for all $1\leqslant i\leqslant k$, the inequalities $D<a_i<2D$, $2D<b_i<3D$, and $3D<c_i<4D$ are fulfilled. Moreover

$$\sum_{i=1}^{k}(a_i+b_i)=\sum_{i=1}^{k}c_i=S.$$

Question. Does there exist a partition of the a_i, b_i, c_i into k triples such that every triple consists of one a-element, one b-element, and one c-element such that the c-element equals the sum of the a-element and the b-element?

Now let an instance of NUMERICAL MATCHING WITH TARGET SUMS be given. From this instance, we will construct a row vector $H^*\in\mathbb{N}^m$ and a column vector $V^*\in\mathbb{N}^n$ with $m=k+S$ and $n=4k$ such that the following holds: There exists a horizontally and vertically convex pattern with row projections H^* and column projections V^*, if and only if the NUMERICAL MATCHING WITH TARGET SUMS instance has a solution. Clearly, this will establish NP-completeness of the reconstruction problem for horizontally and vertically convex patterns.

This time, vector H^* consists of k blocks where the ith block ($i=1,\dots,k$) has c_i+1 entries; the first entry of the ith block is a one, and the remaining c_i entries are twos, i.e.,

$$(1,\underbrace{2,2,2,\dots,2}_{c_1\ \text{times}},1,\underbrace{2,2,2,\dots,2}_{c_2\ \text{times}},\\ \dots,\ 1,\underbrace{2,2,2,\dots,2}_{c_k\ \text{times}}).$$

The vector V^* consists of $2k$ blocks, where the ith block with $1\leqslant i\leqslant k$ equals a_i+1,a_i and where the $(k+i)$th block with $1\leqslant i\leqslant k$ equals b_i,b_i, i.e., V^* is of the form

$$(a_1+1,a_1,\ a_2+1,a_2,\ \dots,\ a_k+1,a_k,\\ b_1,b_1,\ b_2,b_2,\ \dots,\ b_k,b_k).$$

Since the remaining arguments are very similar to those that led to Theorem 3, they are omitted.

Theorem 4. *Reconstructing horizontally and vertically convex patterns from their orthogonal projections is* NP-*complete.*

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