Counting 4-Patterns in Permutations Is Equivalent to Counting 4-Cycles in Graphs

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[1]University of Wrocław

December 14, 2020

Problems concerning permutations

What is the minimal number of swaps of neighboring elements needed to sort a permutation?

$$3 \quad 1 \quad 2 \quad 5 \quad 4$$

Kendall '38

[Kendall's τ rank / bubble-sort] distance counts inversions.

Is the permutation sortable with a stack?

Knuth '68

π can be sorted by a stack iff π avoids 231 (e.g. 1 $\underline{3}$ $\underline{4}$ 5 $\underline{2}$ can't be) .

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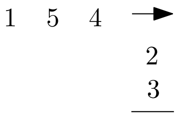

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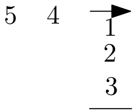

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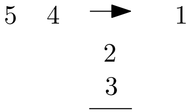

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$5\ 4\ 6 \approx 2\ 1\ 3$

- 21: inversion (e.g. <u>3</u> 2 <u>1</u>)
- 1234: increasing subsequence (e.g. 8 <u>1</u> 5 <u>3</u> 2 <u>4</u> <u>7</u> 6)

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Year	Authors	Runtime
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2013	Guillemot & Marx	$2^{\mathcal{O}(k^2 \log k)} n$
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Patterns of length $k \leq 3$ can be counted in $\tilde{\mathcal{O}}(n)$ time.

4-patterns: $\mathcal{O}(n^2)$ [HellerH'16, WeihsDL'16, WeihsDM'18]

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- eight 4-patterns can be counted in $\tilde{\mathcal{O}}(n)$ time
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This work

- Can we do better? $\mathcal{O}(n^{1.48})$
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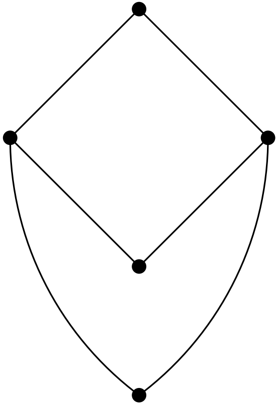
Counting 4-cycles

Input: simple, undirected graph
Output: number of simple cycles of length 4



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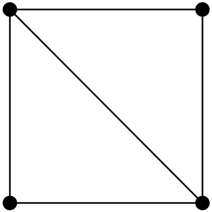

1

History of $2k$-cycles

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		$\mathcal{O}(m^{4/3})$	find a 4-cycle
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Conjecture [Yuster and Zwick, J. Discr. Math.'97]

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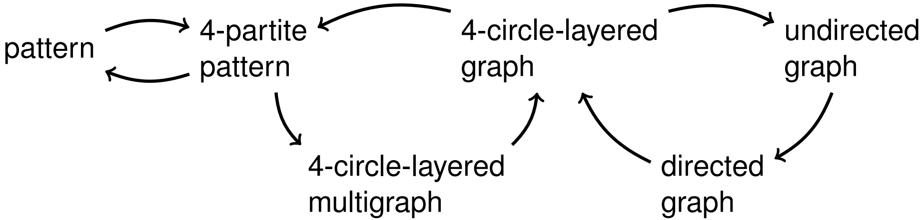
Our contribution

Counting 4-patterns in permutations is equivalent to counting 4-cycles in graphs



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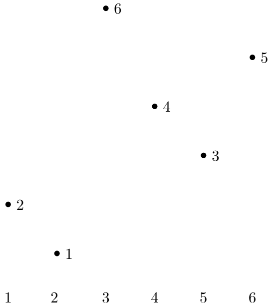
For permutation π, create set of points $S_\pi = \{(i, \pi(i)) : i \in [n]\}$:



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  $1423 \implies 3241$
- vertical reflection ($\updownarrow$) replaces element x with $(n+1) - x$:
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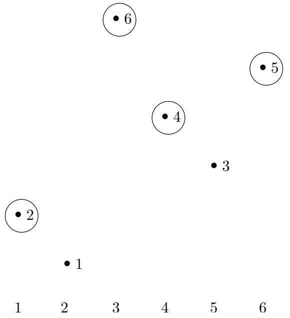

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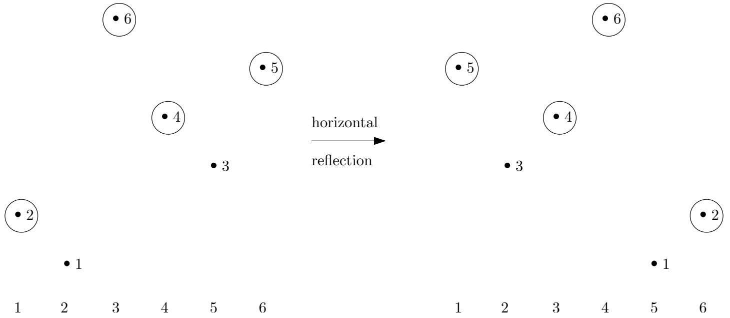

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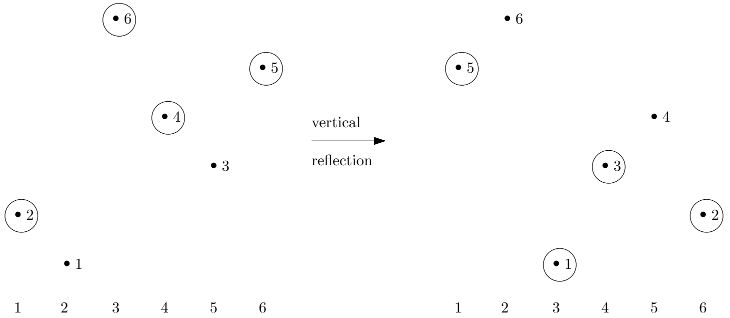

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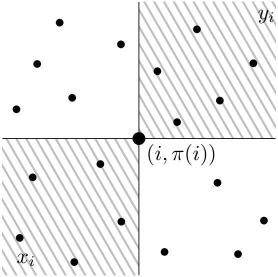
Warm-up: counting 123 and 132

How many occurrences of 123 have middle element at position *i*?



x_i, y_i can be retrieved with orthogonal range queries in $\mathcal{O}(\log n)$ time.

$$\#_{123}(\pi) = \sum_{i=1}^{n} x_i \cdot y_i$$

and then: $\#_{132}(\pi) = \sum_{i=1}^{n} \binom{y_i}{2} - \#_{123}(\pi)$

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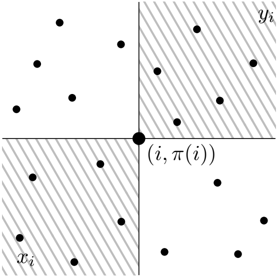

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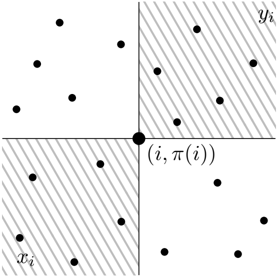

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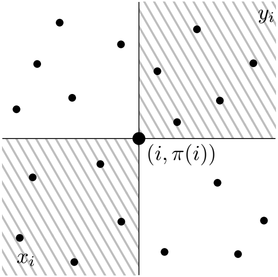

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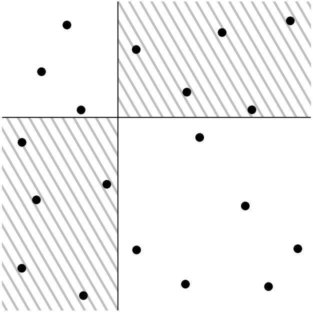
Let's make our lives simpler

Consider a division of the plane into quadrants:



How many occurrences of 1324 look like $\frac{0|3}{1|0}$? $|BL| \cdot \#_{213}(TR)$
How many occurrences of 2134 look like $\frac{0|2}{2|0}$? $\#_{21}(BL) \cdot \#_{12}(TR)$

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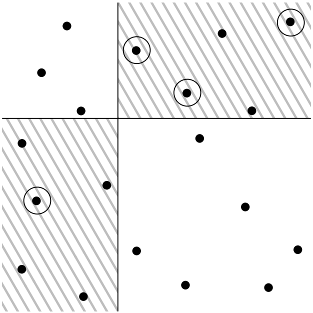

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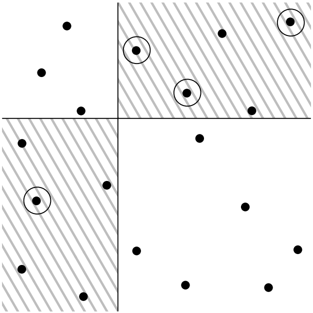

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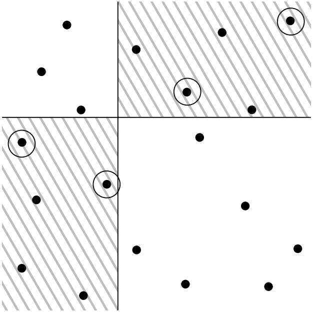

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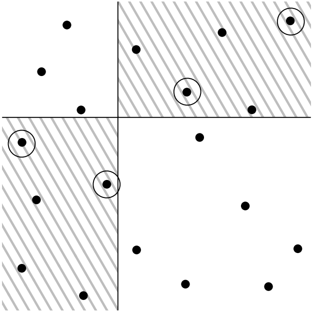

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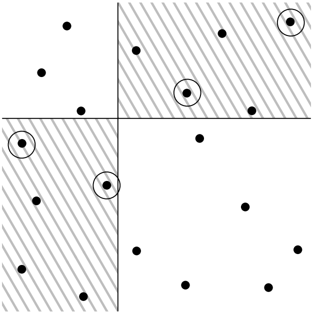

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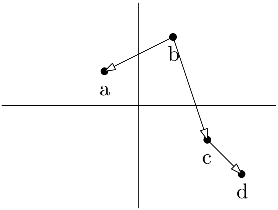

After preprocessing of BR, we count triples (a, c, d) that together with b form 3421 in $\mathcal{O}(\log n)$ time.

With similar techniques we can count patterns that look like $\frac{1|2}{0|1}$.

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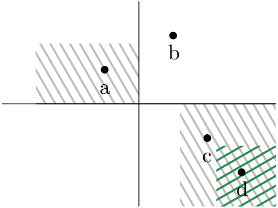

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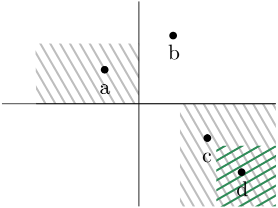

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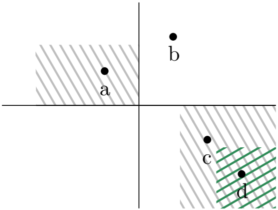

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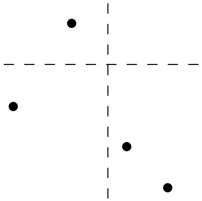

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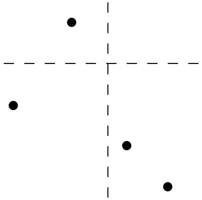

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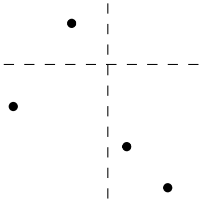

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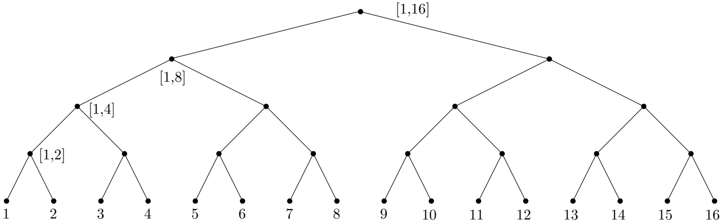

Observation: every element belongs to $\log n$ base ranges.

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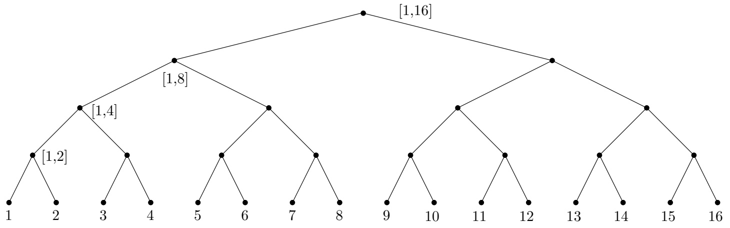

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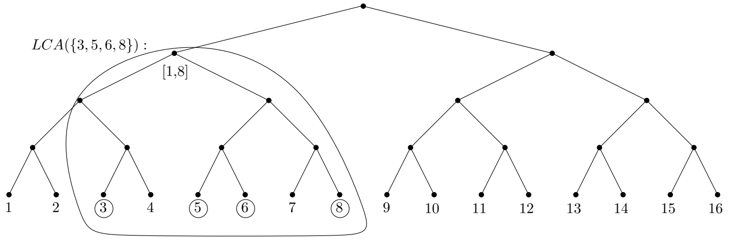

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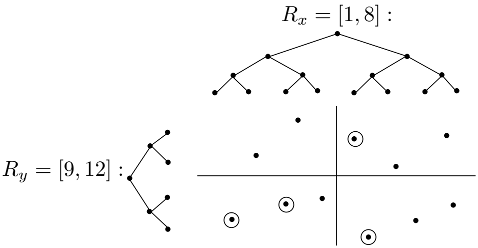

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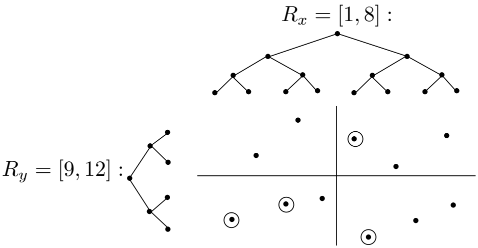

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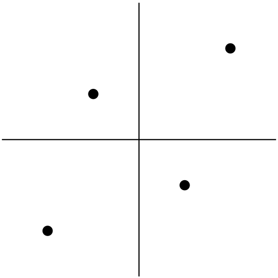

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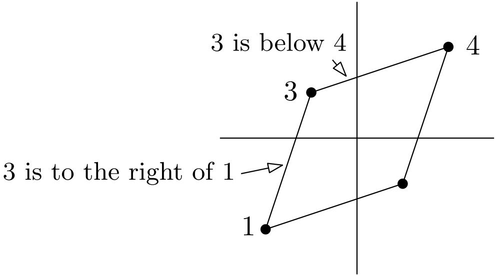

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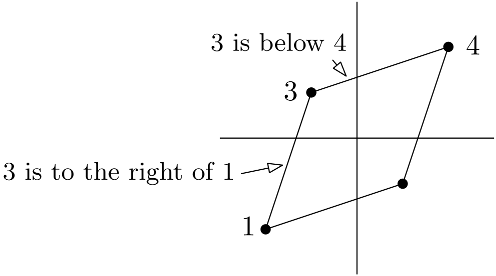

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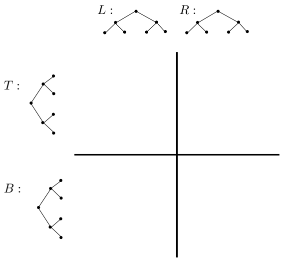
Build full binary tree on elements from each **half-axis**.



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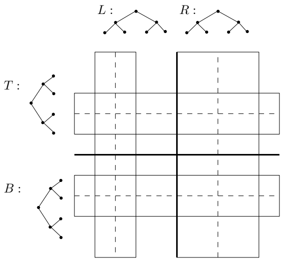

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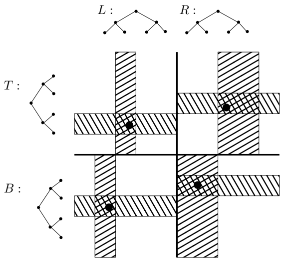

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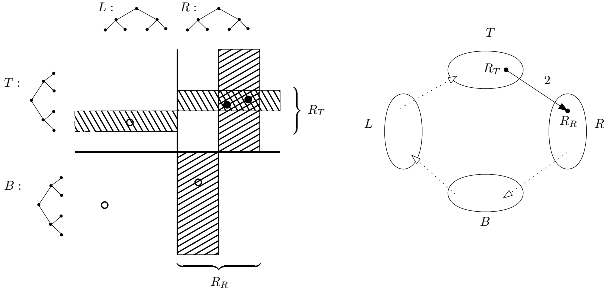

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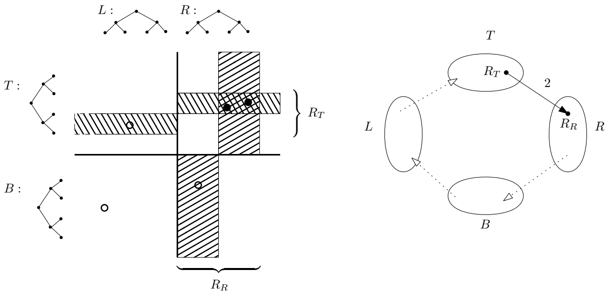

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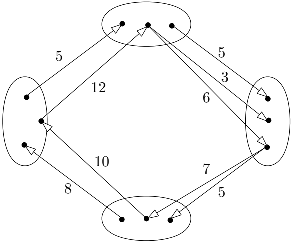

Guess the power of 2 on each side of the graph and multiply by $2^{\sum p_i}$.

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We reduced counting a pattern on n points to a number of instances of counting C_4 in 4-circle-layered graphs that have in total $\tilde{O}(n)$ edges.

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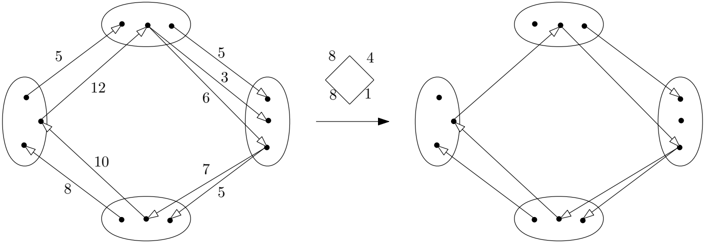

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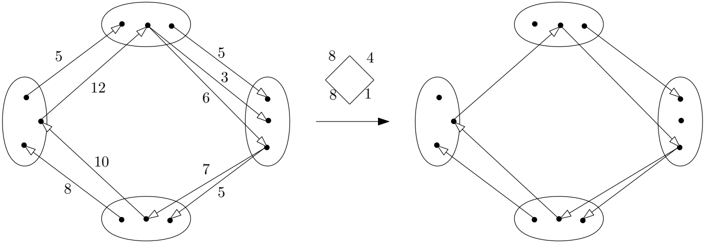

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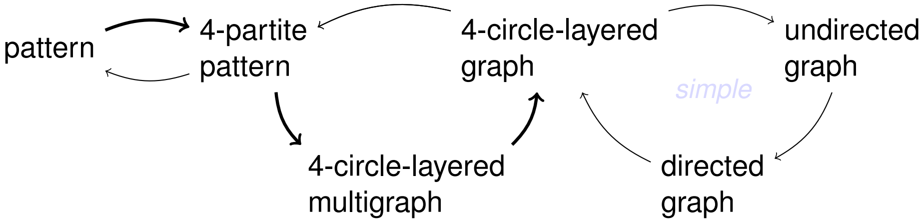

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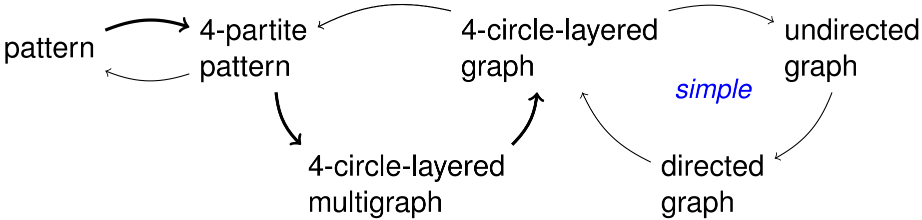

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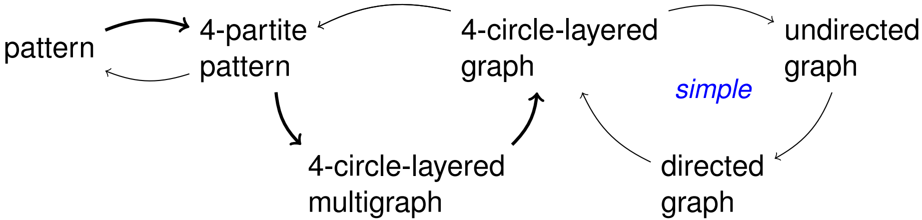

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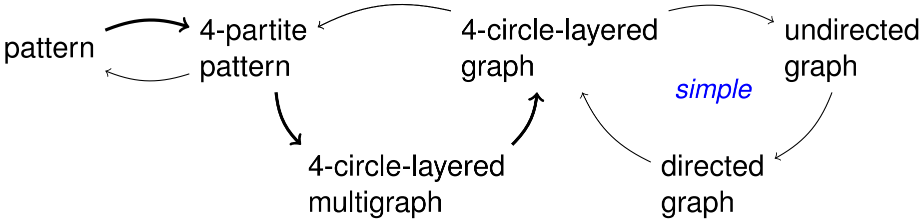

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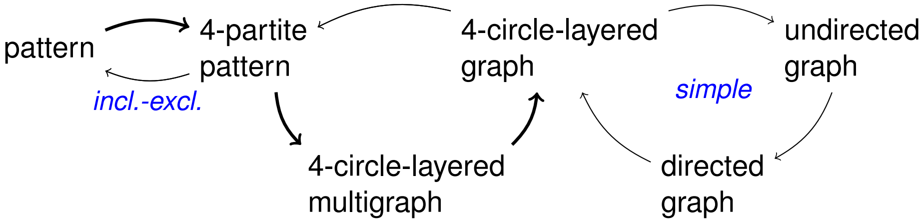

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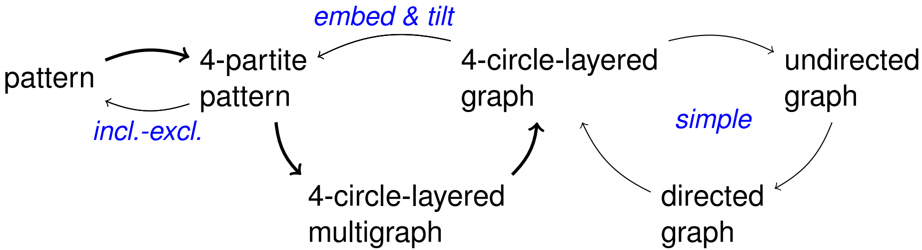

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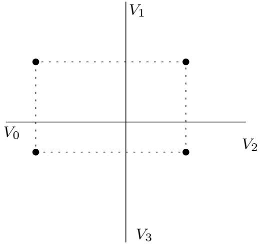

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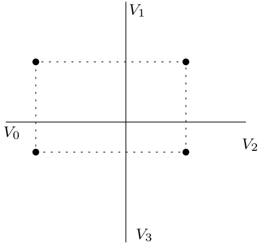

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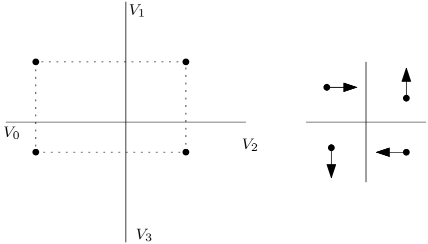

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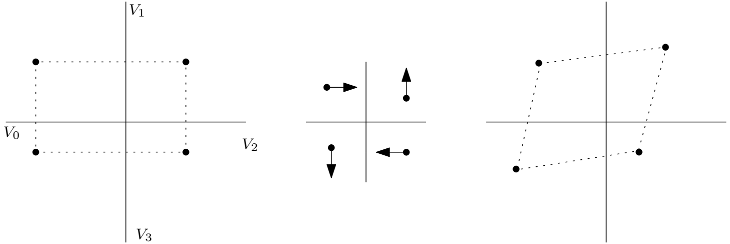

1. Embed 4-circle-layered graph in the plane
   $\implies$ every 4-cycle corresponds to a rectangle
2. Tilt each quadrant separately
   $\implies$ every 4-cycle corresponds to an occurrence of 1324_4
3. Make sure coordinates are distinct and subtract extra occurrences of 1324_4

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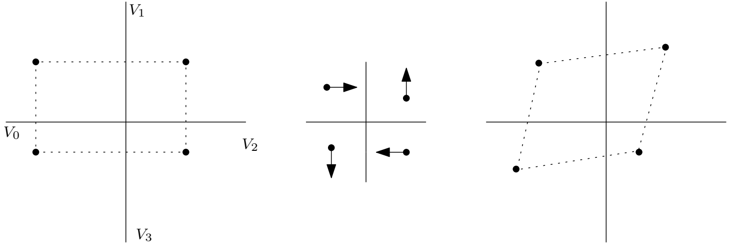

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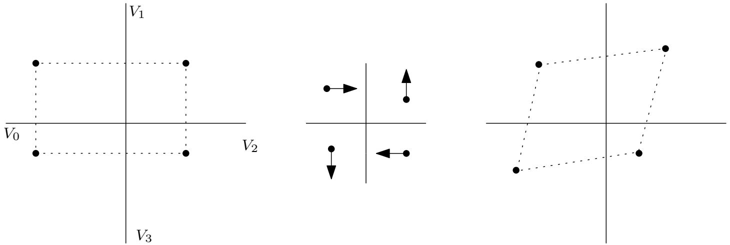

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Summary

Theorem

An $\mathcal{O}(n^\delta)$-time algorithm for counting 4-patterns gives $\mathcal{O}(m^\delta)$-time algorithm for counting 4-cycles in simple graphs.

Recall:

Conjecture [Dahlgaard et al., STOC'17]

For every $\varepsilon > 0$ no algorithm detects 4-cycles in $\mathcal{O}(m^{4/3-\varepsilon})$ time.

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Berendsohn et al. left the following questions open:

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