9.8 Chaos and Strange Attractors: The Lorenz Equations

In principle, the methods described in this chapter for second order autonomous systems can also be applied to higher order systems as well. In practice, there are several difficulties that arise in trying to do this. One problem is that there is simply a greater number of possible cases that can occur, and the number increases with the order of the system (and the dimension of the phase space). Another is the difficulty of graphing trajectories accurately in a phase space of higher than two dimensions; even in three dimensions it may not be easy to construct a clear and understandable plot of the trajectories, and it becomes more difficult as the number of variables increases. Finally, and this has only been clearly realized in the last few years, there are different and very complex phenomena that can occur, and do frequently occur, in systems of third and higher order that are not present in second order systems. Our goal in this section is to provide a brief introduction to some of these phenomena by discussing one particular third order autonomous system that has been intensively studied in recent years. In some respects the presentation here is similar to the treatment of the logistic difference equation in Section 2.9.

An important problem in meteorology, and in other applications of fluid dynamics, concerns the motion of a layer of fluid, such as the earth's atmosphere, that is warmer at the bottom than at the top; see Figure 9.8.1. If the vertical temperature difference ΔT is small, then there is a linear variation of temperature with altitude, but no significant motion of the fluid layer. However, if ΔT is large enough, then the warmer air rises, displacing the cooler air above it, and a steady convective motion results. If the temperature difference increases further, then eventually the steady convective flow breaks up and a more complex and turbulent motion ensues.

While investigating this phenomenon, Edward N. Lorenz[7] was led (by a process too involved to describe here) to the nonlinear autonomous third order system

$$\begin{aligned} dx/dt &= \sigma(-x + y), \\ dy/dt &= rx - y - xz, \\ dz/dt &= -bz + xy. \end{aligned} \tag{1}$$

Equations (1) are now commonly referred to as the Lorenz equations[8]. Observe that the second and third equations involve quadratic nonlinearities. However, except for being

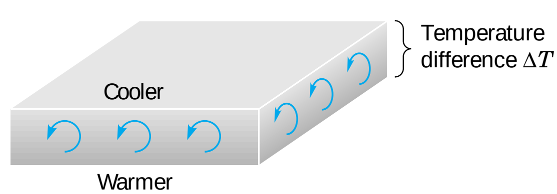

FIGURE 9.8.1 A layer of fluid heated from below.

[7] Edward N. Lorenz (1917–), American meteorologist, received his Ph.D. from the Massachusetts Institute of Technology in 1948 and has been associated with that institution throughout his scientific career. The Lorenz equations were first studied by him in a famous paper published in 1963 dealing with the stability of fluid flows in the atmosphere.

[8] A very thorough treatment of the Lorenz equations appears in the book by Sparrow listed in the references at the end of the chapter.

a third order system, superficially the Lorenz equations appear no more complicated than the competing species or predator–prey equations discussed in Sections 9.4 and 9.5. The variable x in Eqs. (1) is related to the intensity of the fluid motion, while the variables y and z are related to the temperature variations in the horizontal and vertical directions. The Lorenz equations also involve three parameters σ, r, and b, all of which are real and positive. The parameters σ and b depend on the material and geometrical properties of the fluid layer. For the earth's atmosphere reasonable values of these parameters are $\sigma = 10$ and $b = 8/3$; they will be assigned these values in much of what follows in this section. The parameter r, on the other hand, is proportional to the temperature difference ΔT, and our purpose is to investigate how the nature of the solutions of Eqs. (1) changes with r.

The first step in analyzing the Lorenz equations is to locate the critical points by solving the algebraic system

$$\begin{aligned} \sigma x - \sigma y &= 0, \\ rx - y - xz &= 0, \\ -bz + xy &= 0. \end{aligned} \tag{2}$$

From the first equation we have $y = x$. Then, eliminating y from the second and third equations, we obtain

$$x(r - 1 - z) = 0, \tag{3}$$
$$-bz + x^2 = 0. \tag{4}$$

One way to satisfy Eq. (3) is to choose $x = 0$. Then it follows that $y = 0$ and, from Eq. (4), $z = 0$. Alternatively, we can satisfy Eq. (3) by choosing $z = r - 1$. Then Eq. (4) requires that $x = \pm\sqrt{b(r-1)}$ and then $y = \pm\sqrt{b(r-1)}$ also. Observe that these expressions for x and y are real only when $r \geq 1$. Thus (0, 0, 0), which we will denote by P_1, is a critical point for all values of r, and it is the only critical point for $r < 1$. However, when $r > 1$, there are also two other critical points, namely, $(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ and $(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$. We will denote the latter two points by P_2 and P_3, respectively. Note that all three critical points coincide when $r = 1$. As r increases through the value 1, the critical point P_1 at the origin *bifurcates* and the critical points P_2 and P_3 come into existence.

Next we will determine the local behavior of solutions in the neighborhood of each critical point. Although much of the following analysis can be carried out for arbitrary values of σ and b, we will simplify our work by using the values $\sigma = 10$ and $b = 8/3$. Near the origin (the critical point P_1) the approximating linear system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -10 & 10 & 0 \\ r & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{5}$$

The eigenvalues are determined from the equation

$$\begin{vmatrix} -10-\lambda & 10 & 0 \\ r & -1-\lambda & 0 \\ 0 & 0 & -8/3-\lambda \end{vmatrix} = -(8/3+\lambda)[\lambda^2 + 11\lambda - 10(r-1)] = 0. \tag{6}$$

Therefore

$$\lambda_1 = -\frac{8}{3}, \qquad \lambda_2 = \frac{-11 - \sqrt{81 + 40r}}{2}, \qquad \lambda_3 = \frac{-11 + \sqrt{81 + 40r}}{2}. \tag{7}$$

Note that all three eigenvalues are negative for $r < 1$; for example, when $r = 1/2$, the eigenvalues are $\lambda_1 = -8/3$, $\lambda_2 = -10.52494$, $\lambda_3 = -0.47506$. Hence the origin is asymptotically stable for this range of r both for the linear approximation (5) and for the original system (1). However, λ_3 changes sign when $r = 1$ and is positive for $r > 1$. The value $r = 1$ corresponds to the initiation of convective flow in the physical problem described earlier. The origin is unstable for $r > 1$; all solutions starting near the origin tend to grow except for those lying precisely in the plane determined by the eigenvectors associated with λ_1 and λ_2 [or, for the nonlinear system (1), in a certain surface tangent to this plane at the origin].

Next let us consider the neighborhood of the critical point $P_2(\sqrt{8(r-1)/3}, \sqrt{8(r-1)/3}, r-1)$ for $r > 1$. If u, v, and w are the perturbations from the critical point in the x, y, and z directions, respectively, then the approximating linear system is

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}' = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -\sqrt{8(r-1)/3} \\ \sqrt{8(r-1)/3} & \sqrt{8(r-1)/3} & -8/3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{8}$$

The eigenvalues of the coefficient matrix of Eq. (8) are determined from the equation

$$3\lambda^3 + 41\lambda^2 + 8(r+10)\lambda + 160(r-1) = 0, \tag{9}$$

which is obtained by straightforward algebraic steps that are omitted here. The solutions of Eq. (9) depend on r in the following way:

For $1 < r < r_1 \cong 1.3456$ there are three negative real eigenvalues.

For $r_1 < r < r_2 \cong 24.737$ there are one negative real eigenvalue and two complex eigenvalues with negative real part.

For $r_2 < r$ there are one negative real eigenvalue and two complex eigenvalues with positive real part.

The same results are obtained for the critical point P_3. Thus there are several different situations.

For $0 < r < 1$ the only critical point is P_1 and it is asymptotically stable. All solutions approach this point (the origin) as $t \to \infty$.

For $1 < r < r_1$ the critical points P_2 and P_3 are asymptotically stable and P_1 is unstable. All nearby solutions approach one or the other of the points P_2 and P_3 exponentially.

For $r_1 < r < r_2$ the critical points P_2 and P_3 are asymptotically stable and P_1 is unstable. All nearby solutions approach one or the other of the points P_2 and P_3; most of them spiral inward to the critical point.

For $r_2 < r$ all three critical points are unstable. Most solutions near P_2 or P_3 spiral away from the critical point.

However, this is by no means the end of the story. Let us consider solutions for r somewhat greater than r_2. In this case P_1 has one positive eigenvalue and each of P_2 and P_3 has a pair of complex eigenvalues with positive real part. A trajectory can approach any one of the critical points only on certain highly restricted paths. The slightest deviation from these paths causes the trajectory to depart from the critical point. Since none of the critical points is stable, one might expect that most trajectories will approach infinity for large t. However, it can be shown that all solutions remain bounded as $t \to \infty$; see Problem 5. In fact, it can be shown that all solutions ultimately approach a certain limiting set of points that has zero volume. Indeed, this is true not only for $r > r_2$ but for all positive values of r.

A plot of computed values of x versus t for a typical solution with $r > r_2$ is shown in Figure 9.8.2. Note that the solution oscillates back and forth between positive and negative values in a rather erratic manner. Indeed, the graph of x versus t resembles a random vibration, although the Lorenz equations are entirely deterministic and the solution is completely determined by the initial conditions. Nevertheless, the solution also exhibits a certain *regularity* in that the frequency and amplitude of the oscillations are essentially constant in time.

The solutions of the Lorenz equations are also extremely sensitive to perturbations in the initial conditions. Figure 9.8.3 shows the graphs of computed values of x versus t

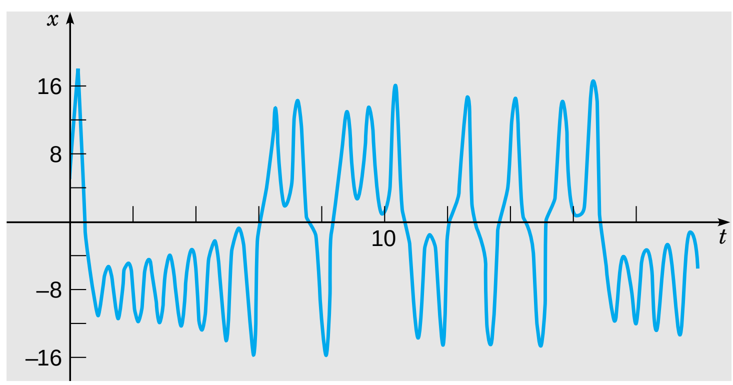

FIGURE 9.8.2 A plot of x versus t for the Lorenz equations (1) with $r = 28$; initial point is (5, 5, 5).

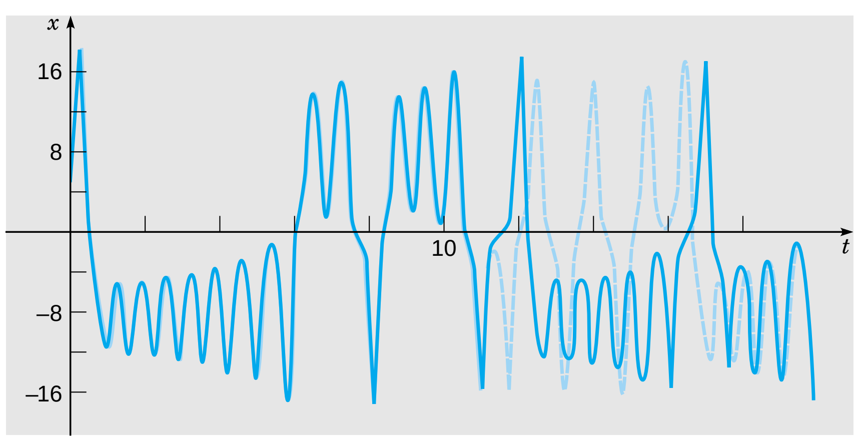

FIGURE 9.8.3 Plots of x versus t for two initially nearby solutions of Lorenz equations with $r = 28$; initial point is (5, 5, 5) for dashed curve and (5.01, 5, 5) for solid curve.

for the two solutions whose initial points are (5, 5, 5) and (5.01, 5, 5). The dashed graph is the same as the one in Figure 9.8.2, while the solid graph starts at a nearby point. The two solutions remain close until t is near 10, after which they are quite different and, indeed, seem to have no relation to each other. It was this property that particularly attracted the attention of Lorenz in his original study of these equations, and caused him to conclude that detailed long-range weather predictions are probably not possible.

The attracting set in this case, although of zero volume, has a rather complicated structure and is called a **strange attractor**. The term **chaotic** has come into general use to describe solutions such as those shown in Figures 9.8.2 and 9.8.3.

To determine how and when the strange attractor is created it is illuminating to investigate solutions for smaller values of r. For $r = 21$ solutions starting at three different initial points are shown in Figure 9.8.4. For the initial point (3, 8, 0) the solution begins to converge to the point P_3 almost at once; see Figure 9.8.4*a*. For the second initial point (5, 5, 5) there is a fairly short interval of transient behavior, after which the solution converges to P_2; see Figure 9.8.4*b*. However, as shown in Figure 9.8.4*c*, for the third initial point (5, 5, 10) there is a much longer interval of transient chaotic behavior before the solution eventually converges to P_2. As r increases, the duration of the chaotic transient behavior also increases. When $r = r_3 \cong 24.06$, the chaotic transients appear to continue indefinitely and the strange attractor comes into being.

One can also show the trajectories of the Lorenz equations in the three-dimensional phase space, or at least projections of them in various planes. Figures 9.8.5 and 9.8.6

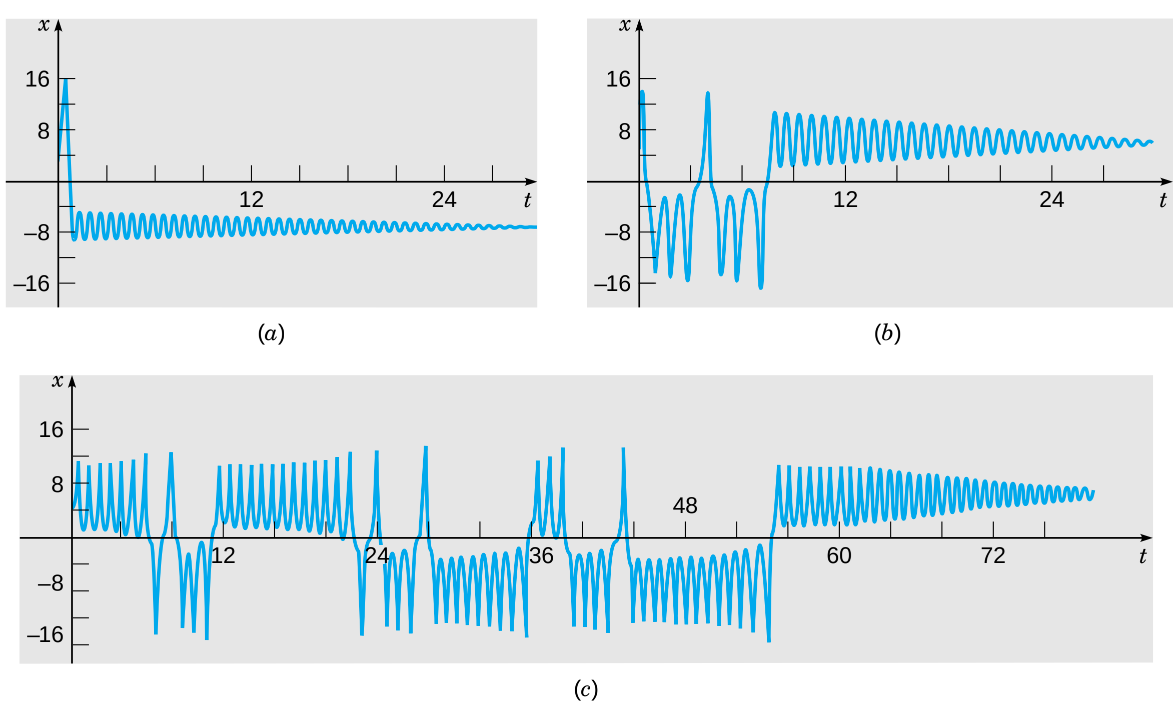

FIGURE 9.8.4 Plots of x versus t for three solutions of Lorenz equations with $r = 21$. (*a*) Initial point is (3, 8, 0). (*b*) Initial point is (5, 5, 5). (*c*) Initial point is (5, 5, 10).

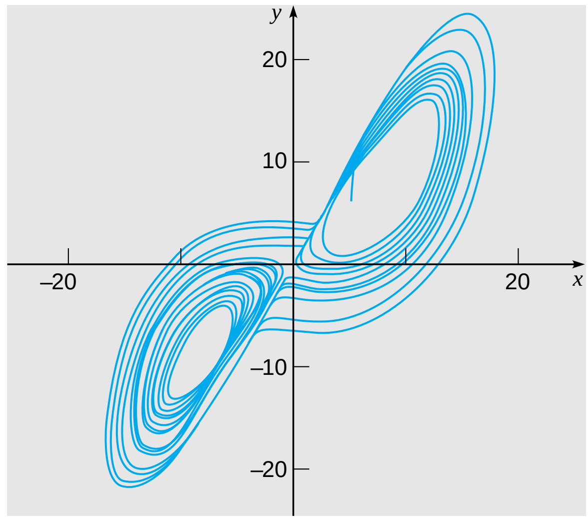

FIGURE 9.8.5 Projections of a trajectory of the Lorenz equations (with $r = 28$) in the xy-plane.

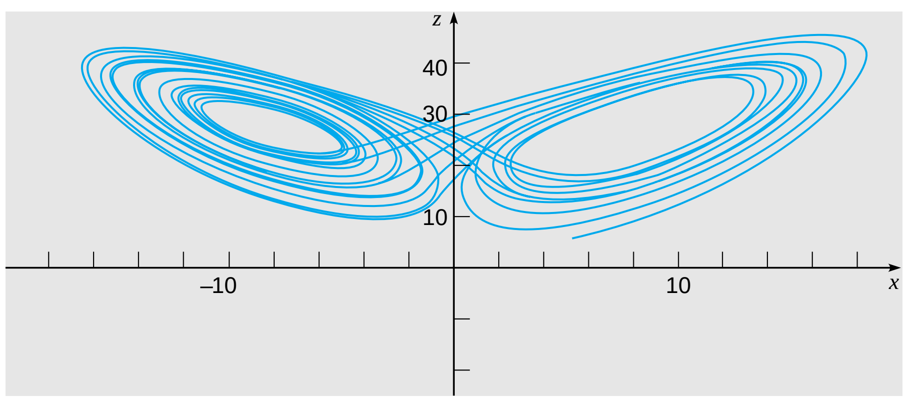

FIGURE 9.8.6 Projections of a trajectory of the Lorenz equations (with $r = 28$) in the xz-plane.

show projections in the xy- and xz-planes, respectively, of the trajectory starting at $(5, 5, 5)$. Observe that the graphs in these figures appear to cross over themselves repeatedly, but this cannot be true for the actual trajectories in three-dimensional space because of the general uniqueness theorem. The apparent crossings are due wholly to the two-dimensional character of the figures.

The sensitivity of solutions to perturbations of the initial data also has implications for numerical computations, such as those reported here. Different step sizes, different numerical algorithms, or even the execution of the same algorithm on different machines will introduce small differences in the computed solution, which eventually lead to large deviations. For example, the exact sequence of positive and negative loops in the calculated solution depends strongly on the precise numerical algorithm and its implementation, as well as on the initial conditions. However, the general appearance of the solution and the structure of the attracting set is independent of all these factors.

Solutions of the Lorenz equations for other parameter ranges exhibit other interesting types of behavior. For example, for certain values of r greater than r_2, intermittent bursts of chaotic behavior separate long intervals of apparently steady periodic oscillation. For other ranges of r, solutions show the period-doubling property that we saw in Section 2.9 for the logistic difference equation. Some of these features are taken up in the problems.

Since about 1975 the Lorenz equations and other higher order autonomous systems have been studied intensively, and this is one of the most active areas of current mathematical research. Chaotic behavior of solutions appears to be much more common than was suspected at first, and many questions remain unanswered. Some of these are mathematical in nature, while others relate to the physical applications or interpretations of solutions.

PROBLEMS

Problems 1 through 3 ask you to fill in some of the details of the analysis of the Lorenz equations in the text.

1. (a) Show that the eigenvalues of the linear system (5), valid near the origin, are given by Eq. (7).
(b) Determine the corresponding eigenvectors.
(c) Determine the eigenvalues and eigenvectors of the system (5) in the case where $r = 28$.
2. (a) Show that the linear approximation valid near the critical point P_2 is given by Eq. (8).
(b) Show that the eigenvalues of the system (8) satisfy Eq. (9).
(c) For $r = 28$ solve Eq. (9) and thereby determine the eigenvalues of the system (8).
3. (a) By solving Eq. (9) numerically show that the real part of the complex roots changes sign when $r \cong 24.737$.
(b) Show that a cubic polynomial $x^3 + Ax^2 + Bx + C$ has one real zero and two pure imaginary zeros only if $AB = C$.
(c) By applying the result of part (b) to Eq. (9) show that the real part of the complex roots changes sign when $r = 470/19$.
4. Use the Liapunov function $V(x, y, z) = x^2 + \sigma y^2 + \sigma z^2$ to show that the origin is a globally asymptotically stable critical point for the Lorenz equations (1) if $r < 1$.
5. Consider the ellipsoid

$$V(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2 = c > 0.$$

(a) Calculate dV/dt along trajectories of the Lorenz equations (1).
(b) Determine a sufficient condition on c so that every trajectory crossing $V(x, y, z) = c$ is directed inward.
(c) Evaluate the condition found in part (b) for the case $\sigma = 10$, $b = 8/3$, $r = 28$.

In each of Problems 6 through 10 carry out the indicated investigations of the Lorenz equations.

6. For $r = 28$ plot x versus t for the cases shown in Figures 9.8.2 and 9.8.3. Do your graphs agree with those shown in the figures? Recall the discussion of numerical computation in the text.
7. For $r = 28$ plot the projections in the xy- and xz-planes, respectively, of the trajectory starting at the point $(5, 5, 5)$. Do the graphs agree with those in Figures 9.8.5 and 9.8.6?
8. (a) For $r = 21$ plot x versus t for the solutions starting at the initial points $(3, 8, 0)$, $(5, 5, 5)$, and $(5, 5, 10)$. Use a t interval of at least $0 \le t \le 30$. Compare your graphs with those in Figure 9.8.4.