Equations over sets of natural numbers.

Artur Jeż

University of Wroclaw

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Part I

Conjunctive Grammars

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Context-free grammars: Rules of the form

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Conjunctive grammars (Okhotin, 2000) Rules of the form

$$A \to \alpha_1 \& \ldots \& \alpha_m$$

"If w is generated by each α_i, then w is generated by A".

Definition of conjunctive grammars

- Quadruple $G = (\Sigma, N, P, S)$, where $S \in N$ and rules in P are

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  - $L_G(A) = \{w \mid A \Longrightarrow^* w\}$

Part II

Unary alphabet and equations over sets of numbers

The case of a unary alphabet

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Resolved equations over sets of natural numbers.

Equations over sets of numbers

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- $EQ(\cup, \cap, \boxplus)$—sets expressible as least solutions

Questions

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Operations $\{\cup, \boxplus\}$:

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- Least solutions are ultimately periodic.

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Remark

Operations $\{\cup, \boxplus\}$:

- Context-free grammars over an alphabet $\{a\}$.
- Least solutions are ultimately periodic.
- General membership problem: NP-complete

Tool: positional notation

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Example

$$(10^*)_4 = \{4^n \mid n \geqslant 0\}$$

Example of non-periodic solution ($k = 4$)

Solution

$$\begin{aligned} L_1 &= 10^* , \\ L_2 &= 20^* , \\ L_3 &= 30^* , \\ L_{12} &= 120^* . \end{aligned}$$

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Equations

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\begin{aligned}
B_1 &= (B_2 \boxplus B_2 \cap B_1 \boxplus B_3) \cup \{1\} , \\
B_2 &= (B_{12} \boxplus B_2 \cap B_1 \boxplus B_1) \cup \{2\} , \\
B_3 &= (B_{12} \boxplus B_{12} \cap B_1 \boxplus B_2) \cup \{3\} , \\
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Example

For example 10^*, the rule is

$$B_1 = (B_2 \boxplus B_2 \cap B_1 \boxplus B_3) \cup \{1\}$$

So we want to prove that

$$10^* = 20^* \boxplus 20^* \cap 10^* \boxplus 30^* \cup \{1\}$$

Calculations

Rule:

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Proof.

$$20^* \boxplus 20^* \quad = \quad 10^+ \cup 20^* 20^*$$

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□

Remark

Similar proof for $ij0^*$ in base-k notation.

Any regular language

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Information the indices carry:

- *leading symbol i*
- *second leading symbol j*
- *q—the computation of M on the rest of the word*

Equations for $B_{i,j,q}$

Example

$$B_{i,j,q} = \bigcup_{(x,q'):q\in\delta(q',x)} \bigcap_{n=1}^{4} B_{i-1,j+n} \boxplus B_{k-n,x,q'} \cup \ldots$$

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$$\begin{array}{ccccc} & & k-n & x & \overbrace{\ldots}^{\text{state } q'} \\ + & i-1 & j+n & & 00\ldots0 \\ \hline & i & j & & \underbrace{x\ldots\ldots}_{\text{state } q} \end{array}$$

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(one-way real-time cellular automata)

Theorem (Jeż, Okhotin, CSR 2007)

$\forall$ trellis automaton M over Σ_k with $L(M) \subseteq \Sigma_k^* \setminus 0\Sigma_k^*$, set $L(M)$ is in $EQ(\cap, \cup, \boxplus)$.

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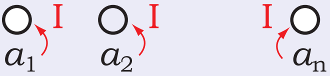

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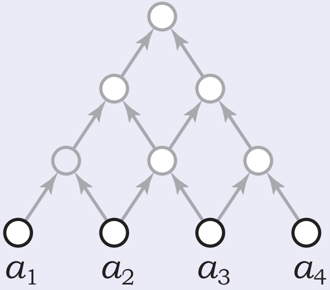

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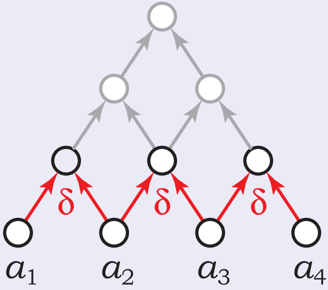

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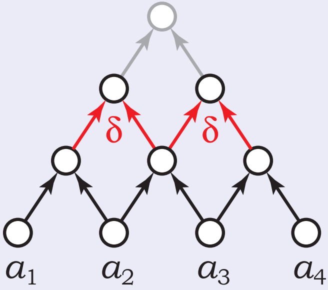

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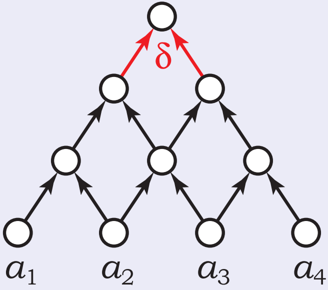

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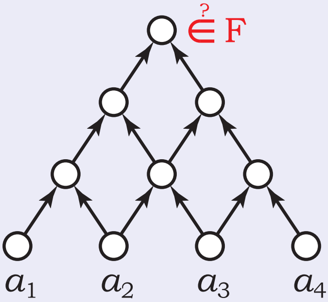

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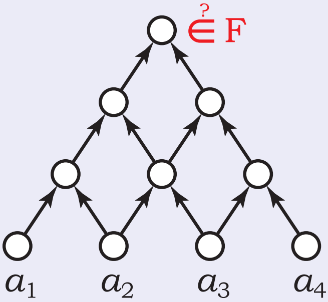

- Closed under $\cup, \cap, \sim$, not closed under concatenation.

Main lemma

Lemma

For every trellis automaton M over Σ_k with $L(M) \subseteq \Sigma_k^* \setminus 0\Sigma_k^*$, there exists a system of equations in $EQ = (\cup, \cap, \boxplus)$ with least solution

$$\{1w10^* \mid w + 1 \in L(M)\}, \ldots,$$

Main lemma

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- $1w10^*$ represents w.

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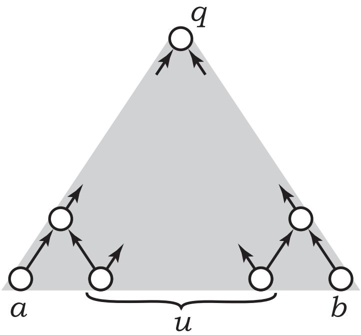

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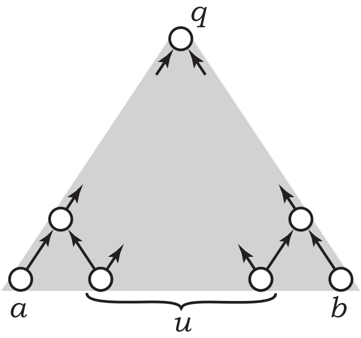

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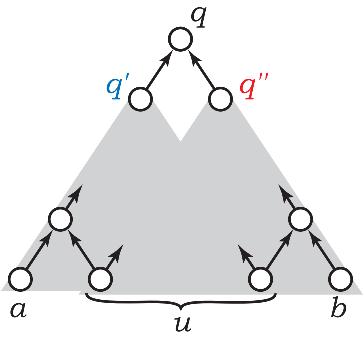

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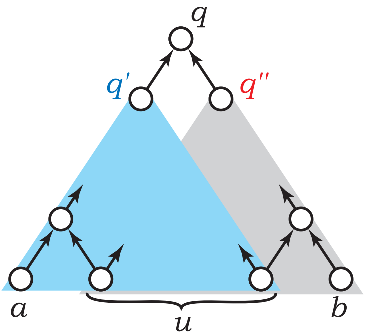

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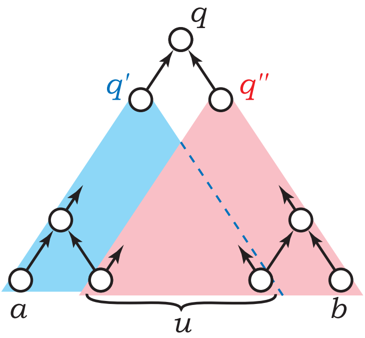

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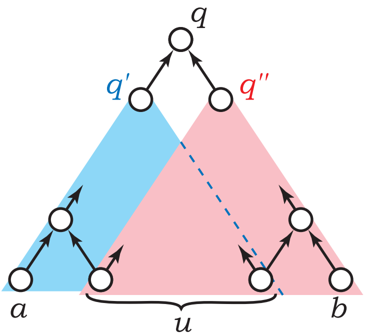

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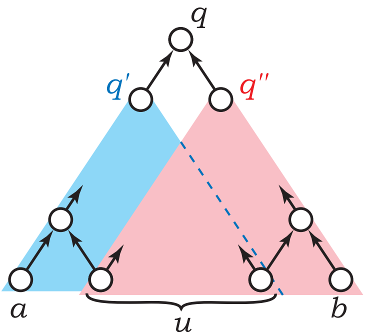

$$\lambda_a(1w10^k) = 1aw10^k$$

$$\rho_b(1w10^k) = 1wb10^{k-1}$$

Part III

Complexity of equations with $\{\cup, \cap, \boxplus\}$

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Theorem (Jeż, Okhotin, STACS 2008)

$EQ(\cup, \cap, \boxplus)$ *contains an EXPTIME-complete set.*

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Theorem (A. Chandra, D. Kozen, L. Stockmeyer 1981)

$$APSPACE = EXPTIME$$
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- *Solution: restricting the model and adding a counter*

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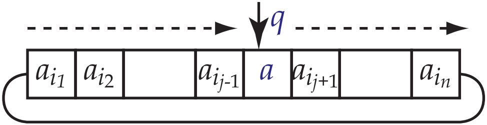

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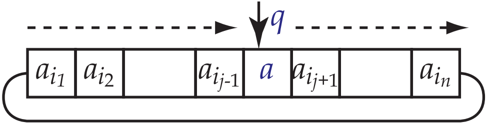

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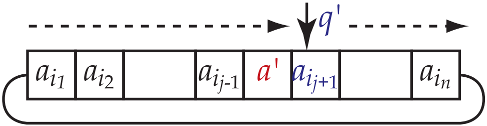

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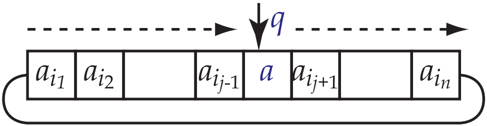

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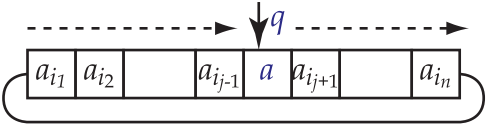

Remark

Still $APSPACE = EXPTIME$.

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$$\underbrace{1 c_{\ell-1} \ldots c_1 c_0}_{\textit{counter}} 55 \underbrace{0\langle a_{i_1} \rangle \ldots 0\langle a_{i_{j-1}} \rangle \langle q \rangle \langle a_{i_j} \rangle 0 \langle a_{i_{j+1}} \rangle \ldots 0 \langle a_{i_n} \rangle 0}_{\textit{tape}} \in \Sigma^*$$

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- Decreases at every step of computation.

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- Equation:

$$\begin{aligned}\text{Move}_{q,a,q',a'}(X) = \big(X \cap \text{Counter}\,55\,\text{Tape}_{q'a'}\big)&\\ \boxplus\big(\langle q\rangle\langle a\rangle 0 \boxminus \langle a'\rangle\langle q'\rangle\big)(00)^*&\\ \cap\,\text{Counter}\,55\,\text{Tape}_{aq}&\end{aligned}$$

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\mathrm{Step}(X) = \Big(\bigcup_{q \in Q_E, a \in \Gamma} \ \bigcup_{(q',a') \in \delta(q,a)} \mathrm{Move}_{q',a',q,a}(X) \Big) \cup
$$

$$
\cup
$$

Constructing equations: upper level

$$X = \mathrm{Final} \cup \mathrm{Step}(X) \cup \big(Y \cap \mathrm{Counter\,55\,Tape}\big)$$
$$Y = \mathrm{Jump}(X) \cup \mathrm{Carry}(Y)$$

- Final: the set of accepting configurations.
- Counter 55 Tape: the set of valid IDs.
- $\mathrm{Step}(X) = \{n \mid \exists m \in X : \ m \vdash n\}$: to the next square.
- $\mathrm{Jump}(X) = \{n \mid \exists m \in X : \ m \vdash' n\}$: to the first symbol.
- $\mathrm{Carry}(X)$: processing the carry in the counter.

$$\mathrm{Step}(X) = \Big(\bigcup_{q \in Q_E, a \in \Gamma} \ \bigcup_{(q',a') \in \delta(q,a)} \mathrm{Move}_{q',a',q,a}(X)\Big) \cup$$
$$\cup \Big(\bigcup_{q \in Q_A, a \in \Gamma} \ \bigcap_{(q',a') \in \delta(q,a)} \mathrm{Move}_{q',a',q,a}(X)\Big)$$

Conclusion

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Problem

Construct a set not representable by equations with $\{\cup, \cap, \boxplus\}$.