Equations over sets of natural numbers

Artur Jeż

Institute of Computer Science
University of Wrocław

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Joint work

Joint work with Alexander Okhotin, University of Turku, Academy of Finland.

Equations over languages

$$\begin{cases} \varphi_1(X_1, \ldots, X_n) & = & \psi_1(X_1, \ldots, X_n) \\ & \vdots & \\ \varphi_m(X_1, \ldots, X_n) & = & \psi_m(X_1, \ldots, X_n) \end{cases}$$

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- Unary languages $\longrightarrow$ numbers
  - resolved
  - unresolved

Numbers and the unary alphabet

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Remark

Focus: resolved (EQ) and unresolved equations over sets of natural numbers with $\cap, \cup, +$.

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$X = \big(X + \{2\}\big) \cup \{0\}$ Unique solution: the even numbers

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5. General: addition only
   Can we use only addition?

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- We focus on properties in base-k notation

Important example—$(10^*)_4$

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Remark

Resolved equations with $\cap, +$ or $\cup, +$ specify only ultimately periodic sets.

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Example (Application)

Let $S \subseteq (10^*\Sigma_k 0^*)_k$. How to obtain
$S' = \{(10^n(d+1)0^m)_k : (10^n d 0^m)_k \in S\}$?

$$S' = \bigcup_{d \in \Sigma_k} \Big(\big(S \cap (10^* d 0^*)_k \big) + (10^*)_k \Big) \cap (10^*(d+1)0^*)_k$$

Application: complexity

Definition

Complexity theory (of a set S)—how many resources are needed to answer a question?
"Given n, does $n \in S$"

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Reduction: Problem $P \geq P'$ if we can answer P (fast) then we can answer P' (fast).

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Given a resolved system with $\cap, \cup, +$ *and a number* n*, does* $n \in S_1$.

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- Using the trick with intersection with regular sets.

More results—greater expressive power

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Theorem

For every k and $R \subset \{0, \ldots, k-1\}^*$ if R is recognised by a trellis automaton M then $(R)_k \in EQ$.

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Definition (Culik, Gruska, Salomaa, 1981)

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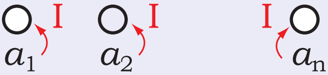

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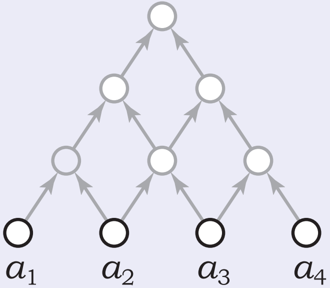

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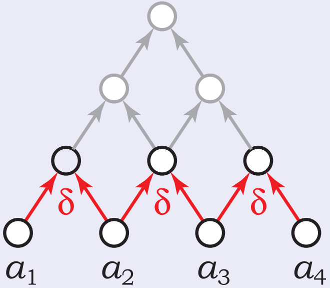

Trellis automata

Definition (Culik, Gruska, Salomaa, 1981)

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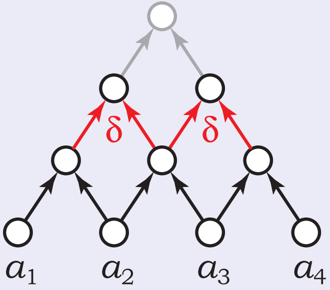

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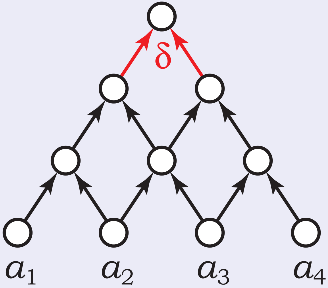

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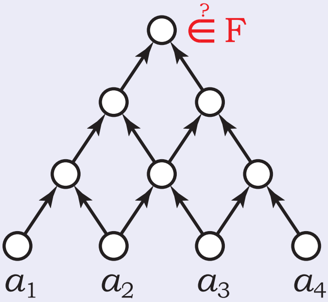

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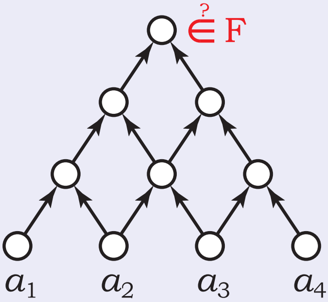

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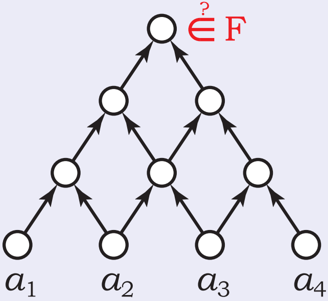

- Closed under $\cup, \cap, \sim$, not closed under concatenation.
- Can recognize $\{wcw\}$, $\{a^n b^n c^n\}$, $\{a^n b^{2^n}\}$, VALC.

Theorem

For every k and $R \subset \{0, \ldots, k-1\}^*$ if R is recognised by a trellis automaton M then $(R)_k \in EQ$.

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✓ Remaking the argument for sets of numbers!

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Remark

Least (greatest) solution—RE-sets (co-RE-sets).

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Problem

How many variables are needed to define something interesting?

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Construct a set not representable by equations with $\{\cup, \cap, +\}$.