Probability & Statistics

Problem set №2. Week starting on March 9th

1. Let Σ be a σ-field on a set Ω.
   (a) Check that $\emptyset \in \Sigma$.
   (b) Suppose $A_k \in \Sigma$, with $k = 1, 2, 3, \ldots$. Prove that $\bigcap_{k\in\mathbb{N}} A_k \in \Sigma$.
2. Let $\Omega = \{a, b, c\}$.
   (a) Describe σ-fields on a set Ω.
   (b) Give examples of functions X, Y such that X is random variable, and Y is not.
3. Let $\Omega = \{1, 2, 3, 4, 5\}$ and $S = \{1, 4\}$. Find the smallest σ-field containing S.
4. Find distribution function (CDF) and expected value $\mathrm{E}(X)$ of the random variable X:

x_i	2	3	4	5
p_i	0.2	0.4	0.1	0.3

5. CDF F of random variable X is given by:

x	$(-\infty; -2]$	$(-2; 3]$	$(3; 5]$	$(5; \infty)$
$F(x)$	0	0.2	0.7	1

   Find density function $f(x)$.
6. Let X be a discrete random variable. Check that $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$.
7. Let X be a continuous random variable. Prove that $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$.
8. **2p.** Check that
   (a) $B(p, q + 1) = B(p, q)\,\dfrac{q}{p + q}$,
   (b) $B(p, q) = B(p, q + 1) + B(p + 1, q)$.
9. **2p.** Prove that $\Gamma(p)\,\Gamma(q) = \Gamma(p + q)\,B(p, q)$, where $p, q \in \mathbb{R}^+$ and > 0 (so all the involved integrals exist).

<u>Def.</u> **Beta function** is the value of the integral

$$B(p, q) = \int_0^1 t^{p-1}\,(1 - t)^{q-1}\,dt, \quad p > 0,\ q > 0.$$

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