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THE HYPERGEOMETRIC FUNCTIONS APPROACH TO THE CONNECTION PROBLEM FOR THE CLASSICAL ORTHOGONAL POLYNOMIALS

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ABSTRACT. Let $\{P_k\}$ and Q_k be any two sequences of classical orthogonal polynomials. Using theorems of the theory of generalized hypergeometric functions, we give closed-form expressions as well as recurrence relations for the coefficients $a_{n,k}$ in the connection equation $Q_n = \sum_{k=0}^{n} a_{n,k} P_k$ $(n \in \mathbb{N})$.

1. Introduction

Let $\{P_k(x)\}$ and $\{Q_k(x)\}$ be two systems of the classical orthogonal polynomials, i. e. associated with the names of Jacobi, Laguerre, Hermite and Bessel. We are looking for the coefficients $a_{n,k}$ in

$$Q_n = \sum_{k=0}^{n} a_{n,k} P_k, \tag{1.1}$$

called *connection coefficients*. An analogous problem is formulated for $\{P_k(x)\}$ and $\{Q_k(x)\}$ being systems of the classical orthogonal polynomials of a discrete variable, i. e. associated with the names of Charlier, Meixner, Krawtchouk and Hahn.

Two type of results are met in a vast literature of this subject: closed-type formulae, or recurrence relations (usually in k) for $a_{n,k}$ (see [2]–[5], [10], [11], [13], [14], [16]–[25], [29]– [33], [36]).

In the present paper, we show that both types of information on $a_{n,k}$ may be obtained using theorems of the theory of generalized hypergeometric functions.

2. Some results on generalized hypergeometric functions

Throughout this section, the letters p, q, r, s, t, u and n stand for non-negative integers. We shall use the Pochhhammer's symbol

$$(\alpha)_k = \Gamma(k+\alpha)/\Gamma(\alpha).$$

The definition for the generalized hypergeometric function is

$${}_pF_q\left(\begin{matrix}[a_p]\\ [b_q]\end{matrix}\,\middle|\, x\right) = \sum_{k=0}^{\infty} \frac{(a_p)_k x^k}{(b_q)_k k!} \tag{2.1}$$

where the symbols $[a_p]$ and $[b_q]$ denote sets $a_1, a_2, \ldots, a_p$ and $b_1, b_2, \ldots, b_q$ of complex parameters, respectively, such that $-b_j \notin \mathbb{N}_0$ and $a_i \neq b_j$ for $1 \le i \le p$ and $1 \le j \le q$, and where the above contracted notation will be used troughout the paper

$$f(a_p)=\prod_{i=1}^{p} f(a_i),\qquad f(b_q)=\prod_{j=1}^{q} f(b_j),$$

f being a given function.

Lemma 2.1.

(2.2)

$$ {}_{p+r+1}F_{q+s}\left(\begin{matrix}-n,\,[a_p],\,[c_r]\\ [b_q],\,[d_s]\end{matrix}\,\middle|\, z\omega\right)=\sum_{k=0}^{n}\binom{n}{k}\frac{(a_p)_k(\alpha_t)_k z^k}{(b_q)_k(\beta_u)_k(k+\lambda)_k}$$
$$\times\ {}_{p+t+1}F_{q+u+1}\left(\begin{matrix}k-n,\,[k+a_p],\,[k+\alpha_t]\\ 2k+\lambda+1,\,[k+b_q],\,[k+\beta_u]\end{matrix}\,\middle|\, z\right)\ {}_{r+u+2}F_{s+t}\left(\begin{matrix}-k,\,k+\lambda,\,[c_r],[\beta_u]\\ [d_s],\,[\alpha_t]\end{matrix}\,\middle|\, \omega\right).$$

Proof. Eq. (2.2) is a specialized form of a much more general result in [26, Vol. II, §9.1, Eq. (13)]. □

Lemma 2.2.

$$ {}_{p+r+1}F_{q+s}\left(\begin{matrix}-n,\,[a_p],\,[c_r]\\ [b_q],\,[d_s]\end{matrix}\,\middle|\, z\omega\right)=\sum_{k=0}^{n}\binom{n}{k}\frac{(a_p)_k(\alpha_t)_k z^k}{(b_q)_k(\beta_u)_k}\tag{2.3}$$
$$\times\ {}_{p+t+1}F_{q+u}\left(\begin{matrix}k-n,\,[k+a_p],\,[k+\alpha_t]\\ [k+b_q],\,[k+\beta_u]\end{matrix}\,\middle|\, z\right)\ {}_{r+u+1}F_{s+t}\left(\begin{matrix}-k,\,[c_r],[\beta_u]\\ [d_s],\,[\alpha_t]\end{matrix}\,\middle|\, \omega\right).$$

Proof. Eq. (2.3) is a specialized form of a much more general result in [26, Vol. II, §9.1, Eq. (27)]. □

Lemma 2.3 (Wimp [34]; see also Wimp [35, App. C]; or Luke [26, Vol. II, §12.4])**.** *Let β, μ, $[\phi_P]$, $[\psi_Q]$ be such that none of the quantities $\beta+1$, ϕ_i, ψ_j and μ are negative integers or zero. Let $w\in\mathbb{C}\setminus\{0\}$ and let $\phi_i=\beta+1$ for $i=P+1$. Then the functions*

$$U_k(w)=\frac{(\phi_P)_k}{(k+\mu)_k(\psi_Q)_k}\,w^k\ {}_{P+1}F_{Q+1}\left(\begin{matrix}[k+\phi_{P+1}]\\ 2k+\mu+1,[k+\psi_Q]\end{matrix}\,\middle|\, w\right)\qquad (k\ge 0),\tag{2.4}$$

satisfy the difference equation

$$\sum_{m=0}^{\theta+1}\{A_m(k;\theta+1)w+B_m(k;\theta+1)\}\,U_{k+m}(w)=0,\tag{2.5}$$

where $\theta:=\max(P,Q+1)$,

$$A_0(k;\,R)\equiv 1,\tag{2.6}$$

$$A_m(k;\,R)=(-1)^m\frac{(2k+\mu)_m(k+\beta+1)_m}{m!(k+\mu)_m}\times\ {}_{P+3}F_{P+2}\left(\begin{matrix}-m,\,2k+\mu+m,[k+\phi_{P+1}+1]\\ 2k+\mu+R+1,[k+\phi_{P+1}]\end{matrix}\,\middle|\, 1\right)\quad (1\le m\le R),\tag{2.7}$$

$$B_0(k,\, R) = B_R(k,\, R) \equiv 0, \tag{2.8}$$

$$B_m(k;\, R) = (-1)^m \frac{(2k+\mu)_{m+1}(k+\beta+1)_m[k+\psi_Q]}{(m-1)!(k+\mu)_m[k+\phi_{P+1}]} \times {}_{Q+2}F_{Q+1}\left(\begin{matrix} 1-m,\, 2k+\mu+m+1, [k+\psi_Q+1] \\ 2k+\mu+R+1,\, [k+\psi_Q] \end{matrix}\middle| 1\right) \quad (1 \le m \le R). \tag{2.9}$$

Functions $U_k(w)$ do not satisfy any other difference equation of type (2.5)*, with A_m and B_m independent of w, of order $\le \theta + 1$.*

In the special case of $P = Q+1$ and $w = 1$, the above result can be refined in the following sense.

Lemma 2.4 (Lewanowicz [15])**.** *Functions*

$$U_k(1) = \frac{(\phi_{Q+1})_k}{(k+\mu)_k(\psi_Q)_k}\, {}_{Q+2}F_{Q+1}\left(\begin{matrix} [k+\phi_{Q+2}] \\ 2k+\mu+1,\, [k+\psi_Q] \end{matrix}\middle| 1\right) \tag{2.10}$$

satisfy the difference equation

$$\sum_{m=0}^{\theta} \{A_m(k;\theta) + B_m(k;\theta)\}\, U_{k+m}(1) = 0, \tag{2.11}$$

of order $\theta := Q+1$, notation being that of (2.7)-(2.9)*.*

We have the following confluent version of the result given in Lemma 2.3.

Lemma 2.5. *Let β, $[\phi_P]$, $[\psi_Q]$ be such that none of the quantities $\beta+1$, ϕ_i, ψ_j and μ are negative integers or zero. Let $w \in \mathbb{C}\setminus\{0\}$ and let $\phi_i = \beta+1$ for $i = P+1$. Then the functions*

$$V_k(w) = \frac{(\phi_P)_k}{(\psi_Q)_k}\, w^k\, {}_{P+1}F_Q\left(\begin{matrix} [k+\phi_{P+1}] \\ [k+\psi_Q] \end{matrix}\middle| w\right) \qquad (k \ge 0), \tag{2.12}$$

satisfy the difference equation

$$\sum_{m=0}^{\vartheta} \{C_m(k)w + D_m(k)\}\, V_{k+m}(w) = 0, \tag{2.13}$$

where $\vartheta := \max(P,Q)+1$, $C_0(k) \equiv 1$, $D_0(k) \equiv 0$, and

$$C_m(k) = (-1)^m \frac{(k+\beta+1)_m}{m!}\, {}_{P+2}F_{P+1}\left(\begin{matrix} -m,\, [k+\phi_{P+1}+1] \\ [k+\phi_{P+1}] \end{matrix}\middle| 1\right), \tag{2.14}$$

$$D_m(k) = (-1)^m \frac{(k+\beta+1)_m(k+\psi_Q)}{(m-1)!(k+\phi_{P+1})}\, {}_{Q+1}F_Q\left(\begin{matrix} 1-m,\, [k+\psi_Q+1] \\ [k+\psi_Q] \end{matrix}\middle| 1\right) \tag{2.15}$$

for $m = 1, 2, \ldots, \vartheta$.

Proof. Replacing in (2.5) w by μw and passing to the limit with $\mu \to \infty$, and making use of the confluence principle

$$\lim_{\mu\to\infty} {}_{P+1}F_{Q+1}\left(\begin{matrix} [k+\phi_{P+1}] \\ [k+\psi_Q], 2k+\mu+1 \end{matrix}\middle| \mu w\right) = {}_{P+1}F_Q\left(\begin{matrix} [k+\phi_{P+1}] \\ [k+\psi_Q] \end{matrix}\middle| w\right)$$

(see [26, Vol. I, §3.5]), we obtain the equation

$$\sum_{m=0}^{\theta+1} \{C_m(k)w + D_m(k)\}\, V_{k+m}(w) = 0,$$

with $\theta = \max(P, Q+1)$.

If $P > Q$ then $\theta = P$, and by [26, Vol.I, Eqs. 2.9(14), (15)] $C_{\theta+1}(k) \not\equiv 0$, $D_{\theta+1}(k) \equiv 0$.

If $P \le Q$ then $\theta = Q+1$, and by the above-cited equations $C_{\theta+1}(k) = D_{\theta+1}(k) \equiv 0$, $D_\theta(k) \not\equiv 0$. Hence the result. □

In the special case of $P = Q$ and $w = 1$, the above result can be refined in the following sense.

Lemma 2.6. *The functions*

$$V_k(1) = \frac{(\phi_{Q+1})_k}{(\psi_Q)_k}\, {}_{Q+1}F_Q\left(\begin{matrix}[k+\phi_{Q+1}]\\ [k+\psi_Q]\end{matrix}\,\middle|\, 1\right) \qquad (k \ge 0) \tag{2.16}$$

satisfy the difference equation

$$\sum_{m=0}^{Q} \{C_m(k) + D_m(k)\}\, V_{k+m}(w) = 0, \tag{2.17}$$

notation being that of (2.14)-(2.15) *with* $P = Q$.

Proof. In Lemma 2.5, let $P := Q$ and $w := 1$. It suffices to show that the last term of the sum in (2.13) vanishes, so that the order of the difference equation reduces to Q.

Indeed, using [26, Vol. I, Eq. 2.9(14)], we obtain

$$C_{Q+1} = \frac{(k+\beta+1)_{Q+1}}{[k+\phi_{Q+1}]} = -D_{Q+1}.$$

□

3. Classical polynomials orthogonal on an interval

Let $\{P_k(x)\}$ and $\{Q_k(x)\}$ be two systems of the classical orthogonal polynomials, i.e. associated with the names of Jacobi, Laguerre, Hermite and Bessel. Let us look for the coefficients $a_{n,k}$ in

$$Q_n(c\,x) = \sum_{k=0}^{n} a_{n,k}(c) P_k(x) \qquad (c \ne 0), \tag{3.1}$$

slightly generalizing equation (1.1).

The following hypergeometric series representation of these polynomials is well known:

$$P_n^{(\alpha,\beta)}(x) = (-1)^n \binom{n+\beta}{n}\, {}_2F_1\left(\begin{matrix}-n,\ n+\alpha+\beta+1\\ \beta+1\end{matrix}\,\middle|\, \frac{1+x}{2}\right), \tag{3.2}$$

$$L_n^{\alpha}(x) = (\alpha+1)_n (-1)^n\, {}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix}\,\middle|\, x\right), \tag{3.3}$$

$$H_n(x) = (2x)^n\, {}_2F_0\left(\begin{matrix}-n/2,\ -n/2+1/2\\ —\end{matrix}\,\middle|\, -\frac{1}{x^2}\right), \tag{3.4}$$

$$Y_n^{\alpha}(x) = {}_2F_0\left(\begin{matrix}-n,\ n+\alpha+1\\ —\end{matrix}\,\middle|\, -\frac{x}{2}\right). \tag{3.5}$$

Shifted Jacobi polynomials are given by

$$R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1). \tag{3.6}$$

3.1. **Explicit forms for the connection coefficients.** The above results allow to obtain connection formulae between classical orthogonal polynomials. For instance, the following expansions hold:

$$
\left\{
\begin{aligned}
Y_n^{\beta}(c\,x) &= \sum_{k=0}^{n} a_{n,k}^{BB}(c) Y_k^{\alpha}(x),\\
a_{n,k}^{BB}(c) &= \binom{n}{k} \frac{(n+\beta+1)_k}{(k+\alpha+1)_k}\, c^k\, {}_2F_1\left(\left.\begin{matrix} k-n,\, k+n+\beta+1 \\ 2k+\alpha+2 \end{matrix}\right| c\right);
\end{aligned}
\right. \tag{3.7}
$$

$$
\left\{
\begin{aligned}
L_n^{\gamma}(c\,x) &= \sum_{k=0}^{n} a_{n,k}^{LL}(c) L_k^{\alpha}(x),\\
a_{n,k}^{LL}(c) &= (-1)^n(\gamma+1)_n \binom{n}{k} \frac{(-c)^k}{(\gamma+1)_k}\, {}_2F_1\left(\left.\begin{matrix} k-n,\, k+\alpha+1 \\ k+\gamma+1 \end{matrix}\right| c\right);
\end{aligned}
\right. \tag{3.8}
$$

$$
\left\{
\begin{aligned}
L_n^{\gamma}(c\,x) &= \sum_{k=0}^{n} a_{n,k}^{LB}(c) Y_k^{\alpha}(x),\\
a_{n,k}^{LB}(c) &= (-1)^n(\gamma+1)_n \binom{n}{k} \frac{(\alpha+1)_k(-2c)^k}{(\gamma+1)_k}\, {}_1F_1\left(\left.\begin{matrix} k-n \\ k+\gamma+1 \end{matrix}\right| -2c\right);
\end{aligned}
\right. \tag{3.9}
$$

$$
\left\{
\begin{aligned}
Y_n^{\gamma}(cx) &= \sum_{k=0}^{n} a_{n,k}^{BJ}(c) R_k^{(\alpha,\beta)}(x),\\
a_{n,k}^{BJ}(c) &= \frac{(-n)_k(n+\gamma+1)_k}{(k+\alpha+\beta+1)_k} \left(-\frac{c}{2}\right)^k\\
&\quad \times\, {}_3F_1\left(\left.\begin{matrix} k+\beta+1,\, k-n,\, k+n+\gamma+1 \\ 2k+\alpha+\beta+2 \end{matrix}\right| -\frac{c}{2}\right).
\end{aligned}
\right. \tag{3.10}
$$

$$
\left\{
\begin{aligned}
R_n^{(\gamma,\delta)}(c\,x) &= \sum_{k=0}^{n} a_{n,k}^{JB}(c) Y_k^{\alpha}(x),\\
a_{n,k}^{JB}(c) &= (-1)^n \binom{n}{k} \frac{(k+\delta+1)_{n-k}(n+\gamma+\delta+1)_k}{n!(k+\alpha+1)_k}\, (-2c)^k\\
&\quad \times\, {}_2F_2\left(\left.\begin{matrix} k-n,\, k+n+\gamma+\delta+1 \\ 2k+\alpha+2,\, k+\delta+1 \end{matrix}\right| -2c\right);
\end{aligned}
\right. \tag{3.11}
$$

$$
\left\{
\begin{aligned}
P_n^{(\gamma,\delta)}(x) &= \sum_{k=0}^{n} a_{n,k}^{JJ} P_k^{(\alpha,\beta)}(x),\\
a_{n,k}^{JJ} &= (-1)^n \binom{n+\delta}{n} \frac{(-n)_k(n+\gamma+\delta+1)_k}{(\delta+1)_k(k+\alpha+\beta+1)_k}\\
&\quad \times\, {}_3F_2\left(\left.\begin{matrix} k-n,\, k+\beta+1, k+n+\gamma+\delta+1 \\ 2k+\alpha+\beta+2,\, k+\delta+1 \end{matrix}\right| 1\right);
\end{aligned}
\right. \tag{3.12}
$$

For instance, let us prove (3.12). We use Lemma 2.1. After the following identification of the parameters

$$\begin{aligned} &p = q = t = 1, \quad r = s = u = 0, \\ &[a_p] = \{n+\gamma+\delta+1\}, \quad [b_q] = \{\delta+1\}, \quad [c_r] = [d_s] = \emptyset, \\ &[\alpha_t] = \{\beta+1\}, \quad [\beta_u] = \emptyset, \quad z = 1, \quad \omega = \frac{1+x}{2} \end{aligned}$$

in equation (2.2), we obtain the formula

$$\begin{aligned} {}_2F_1\left(\begin{matrix} -n,\, n+\mu \\ \delta+1 \end{matrix}\,\middle|\, \frac{1+x}{2}\right) &= \sum_{k=0}^{n} \binom{n}{k} \frac{(\beta+1)_k (n+\mu)_k}{(k+\lambda)_k(\delta+1)_k} \\ &\times\ {}_3F_2\left(\begin{matrix} k-n,\, k+\beta+1, k+n+\mu \\ 2k+\lambda+1,\, k+\delta+1 \end{matrix}\,\middle|\, 1\right) {}_2F_1\left(\begin{matrix} -k,\, k+\lambda \\ \beta+1 \end{matrix}\,\middle|\, \frac{1+x}{2}\right) \end{aligned}$$

with $\lambda := \alpha+\beta+1$ and $\mu := \gamma+\delta+1$. Hence the result.

The proof of the remaining formulae is similar.

3.2. **Recurrence relations for the connection coefficients.** On applying Lemmata 2.3-2.6, one can easily obtain a recurrence relation of the form

$$\sum_{i=0}^{r} A_i(k) a_{n,k+i}(c) = 0 \tag{3.13}$$

for the connection coefficients in (3.1). Table 1 contains the values of the order r of this recurrence relation for many pairs of the classical families.

TABLE 1. The order r of the recurrence relation (3.13)

	$P_k(x)$		
$Q_n(cx)$	$L_k^{\alpha}(x)$	$R_k^{(\alpha,\beta)}(x)$	$Y_k^{\alpha}(x)$
$L_n^{\gamma}(cx)$	2 (general case) 1 $(c=1)$	3	2
$R_n^{(\gamma,\delta)}(cx)$	3	3 (general case) 2 $(c=1 \vee \beta=\delta)$ 1 $(c=1 \wedge \beta=\delta)$	3
$Y_n^{\gamma}(cx)$	3	3	2 (general case) 1 $(c=1)$

4. Classical orthogonal polynomials of a discrete variable

Now, let $\{P_k(x)\}$ and $\{Q_k(x)\}$ be two systems of the classical orthogonal polynomials of a discrete variable, i. e. associated with the names of Charlier, Maeixner, Krawtchouk and Hahn. Let us look for the coefficients $a_{n,k}$ in

$$Q_n(x) = \sum_{k=0}^{n} a_{n,k} P_k(x). \tag{4.1}$$

The following hypergeometric series representations of the classical polynomials are well known:

$$Q_n(x;\alpha,\beta,N)(x) = {}_3F_2\left(\begin{matrix}-n,\ -x,\ n+\alpha+\beta+1\\ \alpha+1,\ -N\end{matrix}\middle|1\right), \tag{4.2}$$

$$m_n^{(\gamma,\mu)}(x) = (\gamma)_n\, {}_2F_1\left(\begin{matrix}-n,\ -x\\ \gamma\end{matrix}\middle|1-\frac{1}{\mu}\right), \tag{4.3}$$

$$k_n^{(p)}(x,N) = (-p)^n \binom{N}{n}\, {}_2F_1\left(\begin{matrix}-n,\ -x\\ -N\end{matrix}\middle|-\frac{1}{p}\right), \tag{4.4}$$

$$c_n^{(\mu)}(x) = {}_2F_0\left(\begin{matrix}-n,\ -x\\ —\end{matrix}\middle|-\frac{1}{\mu}\right). \tag{4.5}$$

4.1. **Explicit forms for the connection coefficients.** As in the "continuous" case, we can very easily obtain connection formulae between classical orthogonal polynomials. For instance, the following expansions hold:

$$\begin{cases} Q_n(x;\alpha,\beta,N)(x) = \displaystyle\sum_{k=0}^{n} a_{n,k}^{HM} m_n^{(\gamma,\mu)}(x), \\ a_{n,k}^{HM} = \displaystyle\binom{n}{k} \frac{(n+\alpha+\beta+1)_k}{(1-N)_k(\beta+1)_k}\left(\frac{\mu}{\mu-1}\right)^k \\ \qquad \times\ {}_3F_2\left(\begin{matrix}k+\gamma,\ k-n,\ k+n+\alpha+\beta+1\\ k+\beta+1,\ k+1-N\end{matrix}\middle|\frac{\mu}{\mu-1}\right); \end{cases} \tag{4.6}$$

$$\begin{cases} c_n^{(\mu)}(x) = \displaystyle\sum_{k=0}^{n} a_{n,k}^{CH} Q_k(x;\alpha,\beta,N)(x), \\ a_{n,k}^{CH} = \displaystyle\binom{n}{k} \frac{(1-N)_k(\beta+1)_k}{(k+\alpha+\beta+1)_k(-\mu)^k}\, {}_3F_1\left(\begin{matrix}k+\beta+1,\ k+1-N,\ k-n\\ 2k+\alpha+\beta+2\end{matrix}\middle|-\frac{1}{\mu}\right); \end{cases} \tag{4.7}$$

$$\begin{cases} m_n^{(\gamma,\mu)}(x) = \displaystyle\sum_{k=0}^{n} a_{n,k}^{MM}(c) m_k^{(\alpha,\beta)}(x), \\ a_{n,k}^{MM} = \displaystyle\binom{n}{k} \frac{(\gamma)_n}{(\gamma)_k}\left[\frac{\beta(\mu-1)}{\mu(\beta-1)}\right]^k {}_2F_1\left(\begin{matrix}k+\alpha,\ k-n\\ k+\gamma\end{matrix}\middle|\frac{\beta(\mu-1)}{\mu(\beta-1)}\right) \end{cases} \tag{4.8}$$

4.2. **Recurrence relations for the connection coefficients.** On applying Lemmata 2.3-2.5, one can easily obtain a recurrence relation of the form

$$\sum_{i=0}^{r} A_i(k) a_{n,k+i} = 0 \tag{4.9}$$

for the connection coefficients in (4.1). Table 2 contains the value of the order r of this recurrence relation for all the pairs of the classical families.

TABLE 2. The order r of the recurrence relation (4.9)

$Q_n(x)$	$P_k(x)$: $c_k^{(\mu)}(x)$	$m_k^{(\gamma,\mu)}(x)$	$k_k^{(p)}(x,N)$	$Q_k(x;\alpha,\beta,N)$
$c_n^{(\nu)}(x)$	1	2	2	3
$m_n^{(\delta,\nu)}(x)$	2	2 (general case) 1 $(\delta=\gamma \vee \nu=\mu)$	2	3 (general case) 2 $(\delta=\beta+1)$
$k_n^{(q)}(x,M)$	2	2	2 (general case) 1 $(q=p \vee M=N)$	3 (general case) 2 $(M=N+1)$
$Q_n(x;\eta,\theta,M)$	3	3 (general case) 2 $(\gamma=\eta+1)$	3 (general case) 2 $(M=N+1)$	3 (general case) 2 $(M=N)$ 1 $(M=N \wedge \eta=\alpha)$

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