GENERALIZED BERNSTEIN POLYNOMIALS*

STANISŁAW LEWANOWICZ†, PAWEŁ WOŹNY

ABSTRACT. We introduce polynomials $B_i^n(x;\omega|\,q)$, depending on two parameters q and ω, which generalize classical Bernstein polynomials, discrete Bernstein polynomials defined by Sablonnière, as well as q-Bernstein polynomials introduced by Phillips. Basic properties of the new polynomials are given. Also, formulas relating $B_i^n(x;\omega|\,q)$, big q–Jacobi and q–Hahn (or dual q-Hahn) polynomials are presented.

AMS Subject Classification: 41A10, 65D17, 33D45.

Keywords: Generalized Bernstein polynomials, generalized Bézier form, generalized de Casteljau algorithm, degree elevation, big q-Jacobi polynomials, q-Hahn polynomials, dual q-Hahn polynomials, orthogonal polynomial expansion.

1. INTRODUCTION.

We define *generalized Bernstein polynomials of degree* n ($n \in \mathbb{N}$) by

$$B_i^n(x;\omega|\,q) := \frac{1}{(\omega;\,q)_n}\begin{bmatrix} n \\ i \end{bmatrix}_q x^i\,(\omega x^{-1};\,q)_i\,(x;\,q)_{n-i} \qquad (i=0,1,\ldots,n), \tag{1.1}$$

where q and ω are real parameters such that $q \neq 1$, and $\omega \neq 1, q^{-1}, \ldots, q^{1-n}$. Here we use the *$q$-Pochhammer symbol* defined for any $c \in \mathbb{C}$ by

$$(c;\,q)_0 := 1, \qquad (c;\,q)_k := \prod_{j=0}^{k-1}(1 - c\,q^j) \qquad (k \geq 1),$$

and the *q–binomial coefficient* given by

$$\begin{bmatrix} n \\ i \end{bmatrix}_q := \frac{(q;\,q)_n}{(q;\,q)_i\,(q;\,q)_{n-i}}.$$

For convenience we shall always assume that $q \in (0,\,1)$, unless it is otherwise stated.

Alternative forms of the formula (1.1) are:

$$B_i^n(x;\omega|\,q) = \frac{1}{(\omega;\,q)_n}\begin{bmatrix} n \\ i \end{bmatrix}_q \prod_{j=0}^{i-1}(x - \omega q^j)\prod_{k=0}^{n-i-1}(1 - xq^k) \tag{1.2}$$

$$= q^{\binom{i}{2}}\,\frac{(-\omega)^i}{(\omega;\,q)_n}\begin{bmatrix} n \\ i \end{bmatrix}_q (\omega^{-1}q^{1-i}x;\,q)_i\,(x;\,q)_{n-i}. \tag{1.3}$$

Date: **May 8, 2009**.

*__Corrected and extended version of our paper__ [9].

†Corresponding author.

Notice that the classical *Bernstein polynomials* (see, e.g., [3, p. 66])

$$B_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i} \qquad (0 \le i \le n), \tag{1.4}$$

the *discrete Bernstein polynomials* [14], [15]

$$b_i^n(N,x) = \frac{1}{(-N)_n}\binom{n}{i}(-x)_i(x-N)_{n-i} \qquad (0 \le i \le n \le N;\ N \in \mathbb{N}), \tag{1.5}$$

where the *Pochhammer symbol* $(c)_k$ is defined for any $c \in \mathbb{C}$ by

$$(c)_0 := 1, \qquad (c)_k := c(c+1)\cdots(c+k-1) \qquad (k \ge 1),$$

as well as the *q-Bernstein polynomials*

$$b_i^n(x;q) = \begin{bmatrix} n \\ i \end{bmatrix}_q x^i\ (x;\,q)_{n-i} \qquad (0 \le i \le n), \tag{1.6}$$

recently introduced by Phillips (see [10]–[12]), are limit or particular forms of the polynomials $B_i^n(x;\omega|\,q)$. Namely, we have

$$\lim_{q\uparrow 1} B_i^n(x;\omega|\,q) = B_i^n\!\left(\frac{x-\omega}{1-\omega}\right), \tag{1.7}$$

$$\lim_{q\uparrow 1} B_i^n(q^{-x};q^{-N}|\,q) = b_{n-i}^n(N,x), \tag{1.8}$$

$$B_i^n(x;0|\,q) = b_i^n(x;q). \tag{1.9}$$

There are several possible applications of the polynomials $B_i^n(x;\omega|\,q)$. First, we can approximate given function $f \in C[0,\,1]$ by a two-parameter family of polynomials

$$\mathcal{B}_n^{\omega,q}(f;x) := \sum_{i=0}^{n} f\left(\frac{[\,i\,]_q}{[\,n\,]_q}\right) B_i^n(u;\omega|\,q) \qquad (u = \omega + (1-\omega)x;\ 0 \le q,\omega < 1),$$

where we use the notation (1.10). The linear operator mapping f to $\mathcal{B}_n^{\omega,q}(f;\cdot)$ is monotone; moreover, using an argument similar to the one used in [12], one can show that $\mathcal{B}_n^{\omega,q_n}(f;\cdot)$ converges uniformly to f on $[0,\,1]$, provided $0 < q_n < 1$ and $\lim_{n\to\infty} q_n = 1$.

Second, one may define a parametric curve $P_n^{\omega,q}$ (*generalized Bézier curve*, say) by

$$P_n^{\omega,q}(t) = \sum_{i=0}^{n} W_i\, B_i^n(u;\omega|\,q) \qquad (u = \omega + (1-\omega)t;\ 0 \le t \le 1),$$

where $W_i \in \mathbb{R}^d$ ($d \in \{1,2,3\}$, $i = 0,1,\ldots,n$) are given points; this representation, like its previously defined particular forms – Bézier curve [5, Chaper 4] and q-Bézier curve [10], is advantageous for practical computations, on account of its shape preserving property, and the numerical stability of the associated de Casteljau algorithm for curve evaluation (see § 2).

Third, as pointed out in [15], it sometimes needed to represent a polynomial given as a linear combination of orthogonal polynomials in the form of a combination of Bernstein polynomials (discrete, in that case); as a result, a stable algorithm of the polynomial evaluation was obtained. In this connection, we show that coefficients of certain basic hypergeometric orthogonal polynomials, called big q-Jacobi polynomials, in the generalized Bernstein polynomial basis are evaluations of another basic hypergeometric orthogonal polynomials, named dual q-Hahn polynomials. The inverse representation is also given; this time the so-called q-Hahn polynomials are involved.

In § 2, we give a list of basic properties of $B_i^n(x;\omega|\,q)$ such as recurrence and q^{-1}-derivative-recurrence relations, partition of unity, Bézier form of a polynomial, degree elevation, de Casteljau algorithm, and q-Pochhammer polynomial representation, which are analogues of the well-known properties of the classical Bernstein polynomials (see, e.g., [5, Chapter 4]). In § 3, we give an explicit formula relating generalized Bernstein, big q-Jacobi and q-Hahn polynomials (see Thm 3.1), while in § 4, another formula is given, relating big q-Jacobi, generalized Bernstein, and dual q-Hahn polynomials (see Thm 4.1). Also, we show that earlier results on connections between classical Bernstein and Jacobi polynomials ([4], [13]), or discrete Bernstein and Hahn polynomials ([15], [13]), or q-Bernstein and little q-Jacobi polynomials [2] can be easily recovered, using these theorems.

We end this section with a list of notation and terminology used in the paper. For more details the reader is referred to the monographs [1] by G. Andrews, R. Askey and R. Roy, or [6] by G. Gasper and M. Rahman, or the report [7] by R. Koekoek and R. Swarttouw. In the sequel we make use of the convention

$$(c_1,c_2,\ldots,c_k)_n := \prod_{j=1}^{k}(c_j)_n, \qquad (c_1,c_2,\ldots,c_k;\,q)_n := \prod_{j=1}^{k}\,(c_j;\,q)_n.$$

Also, for $c\in\mathbb{C}$ we define the *q–number* $[\,c\,]_q$ by

$$[\,c\,]_q := \frac{q^c-1}{q-1}. \tag{1.10}$$

The q- and q^{-1}-*derivative operators* $\boldsymbol{D}_q$ and $\boldsymbol{D}_{1/q}$ are given by

$$\boldsymbol{D}_q f(x) := \frac{f(qx)-f(x)}{(q-1)x}, \qquad \boldsymbol{D}_{1/q} f(x) := \frac{f(x/q)-f(x)}{(1/q-1)x}, \qquad x\neq 0,$$

and $\boldsymbol{D}_q f(0) := \boldsymbol{D}_{1/q} f(0) := f'(0)$, provided $f'(0)$ exists. Note that

$$\lim_{q\to 1}\boldsymbol{D}_q f(x) = \lim_{q\to 1}\boldsymbol{D}_{1/q} f(x) = f'(x)$$

if f is differentiable. Also, we have

$$\boldsymbol{D}_q^{\,r}\, f(x) = q^{\binom{r}{2}}\boldsymbol{D}_{1/q}^{\,r}\, f(q^r x) \qquad (r=0,1,\ldots). \tag{1.11}$$

The *generalized hypergeometric series* is defined by (see, e.g., [1, § 2.1])

$${}_rF_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\,\middle|\, z\right) := \sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r)_k}{(1,b_1,\ldots,b_s)_k}\,z^k,$$

while the *basic hypergeometric series* is defined by (see, e.g., [1, § 10.9])

$${}_r\phi_s\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{matrix}\,\middle|\, q;z\right) := \sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r;\,q)_k}{(q,b_1,\ldots,b_s;\,q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k,$$

where r, $s\in\mathbb{Z}_+$ and $a_1,a_2,\ldots,a_r$, $b_1,b_2,\ldots,b_s$, $z\in\mathbb{C}$.

2. Properties of the generalized Bernstein polynomials.

Lemma 2.1. *For $n \in \mathbb{N}$ and $0 \le i \le n$ the following holds true (in (iii)–(vi), we adopt the convention that $B_i^n(x;\omega|\,q) = 0$ for $i < 0$ or $i > n$):*

(i) the zeros of $B_i^n(x;\omega|\,q)$ are: $\omega,\ \omega q, \ldots, \omega q^{i-1},\ 1, q^{-1}, \ldots, q^{i-n+1}$;

(ii) $B_i^n(x;\omega|\,q) \ge 0$ for (a) $0 < q < 1$, $0 \le \omega < 1$ and $\omega \le x \le 1$, or (b) $q, \omega > 1$ and $1 \le x \le \omega$;

(iii) $$B_i^n(x;\omega|\,q) = \frac{1 - xq^{n-i-1}}{1-\omega q^{n-1}}\, B_i^{n-1}(x;\omega|q) + q^{n-i}\,\frac{x-\omega q^{i-1}}{1-\omega q^{n-1}}\, B_{i-1}^{n-1}(x;\omega|q);$$

(iv) $$B_i^n(x;\omega|\,q) = q^i\,\frac{1 - xq^{n-i-1}}{1-\omega q^{n-1}}\, B_i^{n-1}(x;\omega|q) + \frac{x-\omega q^{i-1}}{1-\omega q^{n-1}}\, B_{i-1}^{n-1}(x;\omega|\,q);$$

(v) $$B_i^n(x;\omega|\,q) = \frac{[\,n-i+1\,]_q}{[\,n+1\,]_q}\, B_i^{n+1}(x;\omega|\,q) + \left(1 - \frac{[\,n-i\,]_q}{[\,n+1\,]_q}\right) B_{i+1}^{n+1}(x;\omega|\,q);$$

(vi) $$\begin{cases} \boldsymbol{D}_{1/q} B_i^n(x;\omega|\,q) = & q^{-i}\,\dfrac{[\,n\,]_q}{1-\omega}\Big(q\, B_{i-1}^{n-1}(x;q\omega|\,q) - B_i^{n-1}(x;q\omega|\,q)\Big),\\ \boldsymbol{D}_{q} B_i^n(x;\omega|\,q) = & q^{-i}\,\dfrac{[\,n\,]_q}{1-\omega}\Big(q\, B_{i-1}^{n-1}(qx;q\omega|q) - B_i^{n-1}(qx;q\omega|q)\Big); \end{cases}$$

(vii) $$B_i^n(cx;\omega|q) = \sum_{j=i}^{n} B_i^j(c;\omega|q)\, B_j^n(x;\omega/c|q);$$

(viii) $$B_{i+j}^{n+m}(x;\omega|q) = q^{j(i-m)}\,\frac{\begin{bmatrix} n+m \\ i+j \end{bmatrix}_q}{\begin{bmatrix} m \\ i \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q}\, B_i^m(x;\omega|q)\, B_j^n(q^{m-i}x;q^m\omega|q).$$

Proof.
(i), (ii), (viii) These properties follow immediately from the definition (1.1).
(iii), (iv) It is easy to oberve that (cf. (1.1))

$$\frac{[\,i\,]_q}{[\,n\,]_q} B_i^n(x;\omega|\,q) = \frac{x-\omega q^{i-1}}{1-\omega q^{n-1}}\, B_{i-1}^{n-1}(x;\omega|\,q), \tag{2.1}$$

$$\frac{[\,n-i\,]_q}{[\,n\,]_q} B_i^n(x;\omega|\,q) = \frac{1-xq^{n-i-1}}{1-\omega q^{n-1}}\, B_i^{n-1}(x;\omega|\,q). \tag{2.2}$$

By combining these equations and using

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = q^{n-i}\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ i \end{bmatrix}_q = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q + q^i\begin{bmatrix} n-1 \\ i \end{bmatrix}_q,$$

we obtain the stated equations.
(v) Observe that $1 - [\,n-i-1\,]_q/[\,n\,]_q = q^{n-i-1}[\,i+1\,]_q/[\,n\,]_q$, so that (2.1) implies

$$\left(1 - \frac{[\,n-i-1\,]_q}{[\,n\,]_q}\right) B_{i+1}^n(x;\omega|\,q) = q^{n-i-1}\,\frac{x-\omega q^i}{1-\omega q^{n-1}}\, B_i^{n-1}(x;\omega|\,q).$$

By adding the above equation to (2.2), and replacing n by $n+1$ in the resulting equation, we obtain the result.

(vi) We prove the first formula; the second formula follows from the first one by (1.11) with $r=1$. Let us write (cf. (1.3)) $B_i^n(x;\omega|\,q) = C_{n,i}(\omega)\, f(x)\, g(x)$ with

$$C_{n,i}(\omega) = \frac{q^{\binom{i}{2}}(-\omega)^i}{(\omega;\, q)_n}\begin{bmatrix} n \\ i \end{bmatrix}_q, \quad f(x) = (\omega^{-1}q^{1-i}x;\, q)_i, \quad g(x) = (x;\, q)_{n-i}.$$

By using general properties of the q^{-1}-derivative,

$$\boldsymbol{D}_{1/q}\,(\alpha x;\, q)_m = -\alpha\,[\,m\,]_q\,(\alpha x;\, q)_{m-1}, \tag{2.3}$$

$$\boldsymbol{D}_{1/q}(f\,g)(x) = \boldsymbol{D}_{1/q}f(x)\cdot g(x) + f(x/q)\boldsymbol{D}_{1/q}g(x),$$

as well as equations

$$\frac{C_{n,i}(\omega)}{C_{n-1,i-1}(q\omega)} = -\frac{[\,n\,]_q}{[\,i\,]_q}\frac{\omega}{1-\omega}, \qquad \frac{C_{n,i}(\omega)}{C_{n-1,i}(q\omega)} = \frac{[\,n\,]_q}{[\,n-i\,]_q}\frac{q^{-i}}{1-\omega},$$

we obtain

$$\begin{aligned}\boldsymbol{D}_{1/q}B_i^n(x;\omega|\,q) &= -C_{n,i}(\omega)\big([\,i\,]_q\omega^{-1}q^{1-i}\,(\omega^{-1}q^{1-i}x;\, q)_{i-1}\,(x;\, q)_{n-i} \\ &\qquad + [\,n-i\,]_q\,(\omega^{-1}q^{-i}x;\, q)_i\,(x;\, q)_{n-i-1}\big) \\ &= q^{-i}\frac{[\,n\,]_q}{1-\omega}\Big(q\,B_{i-1}^{n-1}(x;q\omega|\,q) - B_i^{n-1}(x;q\omega|\,q)\Big).\end{aligned}$$

□

Lemma 2.2. *For $n\in\mathbb{N}$, the following identities hold:*

$$\sum_{i=0}^{n} B_i^n(x;\omega|\,q) \equiv 1 \qquad (\textit{partition of unity}), \tag{2.4}$$

$$\sum_{i=0}^{n} \frac{[\,i\,]_q}{[\,n\,]_q}B_i^n(x;\omega|\,q) = \frac{x-\omega}{1-\omega}. \tag{2.5}$$

Proof. To prove (2.4), set $a=x$ and $b=\omega/x$ in the identity (see, e.g., [1, § 10, Exercise 9])

$$(ab;\, q)_n = \sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}_q a^i\,(b;\, q)_i\,(a;\, q)_{n-i}.$$

We prove (2.5) by induction on n. For $n=1$, this equation is obviously true. Assume that it holds for a certain $n\in\mathbb{N}$. By using (2.1) and (2.4), we obtain

$$\begin{aligned}\sum_{i=0}^{n+1}\frac{[\,i\,]_q}{[\,n+1\,]_q}B_i^{n+1}(x;\omega|\,q) &= \sum_{i=0}^{n}\frac{x-\omega q^i}{1-\omega q^n}B_i^n(x;\omega|\,q) \\ &= \frac{x-\omega}{1-\omega q^n} + \frac{\omega}{1-\omega q^n}\sum_{i=0}^{n}(1-q^i)B_i^n(x;\omega|\,q) \\ &= \frac{x-\omega}{1-\omega q^n} + \omega\frac{1-q^n}{1-\omega q^n}\sum_{i=0}^{n}\frac{[\,i\,]_q}{[\,n\,]_q}B_i^n(x;\omega|\,q) \\ &= \frac{x-\omega}{1-\omega q^n} + \omega\frac{1-q^n}{1-\omega q^n}\cdot\frac{x-\omega}{1-\omega} = \frac{x-\omega}{1-\omega}.\end{aligned}$$

Hence, the identity holds true for $n = 1, 2, \ldots$. □

Lemma 2.3. *Polynomials $B_0^n(x;\omega|\,q)$, $B_1^n(x;\omega|\,q), \ldots, B_n^n(x;\omega|\,q)$ form a basis in the space Π_n of polynomials of degree $\le n$.*

Proof. The lemma may be easily justified using Lemma 2.9. □

By the above lemma, any polynomial $p \in \Pi_n$ can be written in the *generalized Bézier form*

$$p(x) = \sum_{i=0}^{n} \beta_i \, B_i^n(x;\omega|\,q). \tag{2.6}$$

Lemma 2.4. *For (a) $0 < q < 1$, $0 \le \omega < 1$ and $\omega \le x \le 1$, or (b) $q, \omega > 1$ and $1 \le x \le \omega$, the graph of the polynomial* (2.6) *lies in the convex hull of the points*

$$W_i = \left((1-\omega)\frac{[\,i\,]_q}{[\,n\,]_q} + \omega,\ \beta_i \right) \qquad (i = 0, 1, \ldots, n).$$

Proof. By using Lemma 2.2 we obtain

$$\begin{aligned}(x,\, p(x)) &= \left(\omega + (1-\omega)\frac{x-\omega}{1-\omega},\ p(x) \right)\\ &= \left(\omega \sum_{i=0}^{n} B_i^n(x;\omega|\,q) + (1-\omega)\sum_{i=0}^{n} \frac{[\,i\,]_q}{[\,n\,]_q} B_i^n(x;\omega|\,q),\ \sum_{i=0}^{n} \beta_i\, B_i^n(x;\omega|\,q) \right)\\ &= \sum_{i=0}^{n} B_i^n(x;\omega|\,q)\, W_i.\end{aligned}$$

Hence, in view of part *(ii)* of Lemma 2.1 and (2.4), the point $(x,\, p(x))$ is a convex linear combination of the points $W_0,\, W_1,\, \ldots, W_n$. □

Lemma 2.5. *Let $p \in \Pi_n$ be given in the generalized Bézier form* (2.6)*. Then for $r = 0, 1, \ldots,$ we have*

$$\left\{ \begin{aligned} \boldsymbol{D}_{1/q}^{r}\, p(x) =&\ \frac{\prod_{j=0}^{r-1} [\,n-j\,]_q}{(\omega;\, q)_r} \sum_{i=0}^{n-r} \left(q^{-i}\Delta\right)^r \beta_i \cdot B_i^{n-r}(x; q^r\omega|\,q),\\ \boldsymbol{D}_{q}^{r}\, p(x) =&\ q^{\binom{r}{2}} \frac{\prod_{j=0}^{r-1} [\,n-j\,]_q}{(\omega;\, q)_r} \sum_{i=0}^{n-r} \left(q^{-i}\Delta\right)^r \beta_i \cdot B_i^{n-r}(q^r x; q^r\omega|q), \end{aligned} \right. \tag{2.7}$$

where Δ is the forward progression operator, $\Delta\beta_i = \beta_{i+1} - \beta_i$.

Proof. It suffices to prove the first formula; the second formula follows from the first one by (1.11). For $r = 0$, the first equation (2.7) is trivial. Let us consider the case $r = 1$. By using part *(vi)* of Lemma 2.1, we obtain

$$\begin{aligned}\boldsymbol{D}_{1/q}p(x) &= \sum_{i=0}^{n} \beta_i \boldsymbol{D}_{1/q} B_i^n(x;\omega|\, q)\\ &= \frac{[\,n\,]_q}{1-\omega}\sum_{i=0}^{n} \beta_i q^{-i}\left(q\, B_{i-1}^{n-1}(x;q\omega|\, q) - B_i^{n-1}(x;q\omega|\, q)\right)\\ &= \frac{[\,n\,]_q}{1-\omega}\sum_{i=0}^{n-1}\left(q^{-i}\Delta\right)\beta_i\cdot B_i^{n-1}(x;q\omega|\, q).\end{aligned}$$

Generalization to higher-order q^{-1}-derivatives is straightforward. □

Lemma 2.6. *Let $p \in \Pi_n$ be given in the form* (2.6)*. The q-Pochhammer polynomial expansion of p is given by*

$$p(x) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q (-1)^i \frac{\left(q^{i-n}\Delta\right)^i \beta_{n-i}}{(\omega;\, q)_i}\ (x;\, q)_i. \tag{2.8}$$

Proof. First observe that if $a_0, a_1, \ldots, a_n$ are coefficients in

$$p(x) = \sum_{i=0}^{n} a_i\, (x;\, q)_i,$$

then we have

$$(-1)^i[\,1\,]_q[\,2\,]_q\cdots[\,i\,]_q a_i = \left(\boldsymbol{D}_{1/q}^i p\right)(1) \qquad (i = 0, 1, \ldots, n). \tag{2.9}$$

This can be proved by induction on i, using (2.3). Now, by using Lemma 2.5, and the part *(i)* of Lemma 2.1, we obtain (cf. (2.7))

$$\left(\boldsymbol{D}_{1/q}^i p\right)(1) = \frac{\prod_{j=0}^{i-1}[\,n-j\,]_q}{(\omega;\, q)_i}\left(q^{i-n}\Delta\right)^i \beta_{n-i} \quad (i = 0, 1, \ldots, n).$$

Comparing this formula with (2.9) gives

$$a_i = \begin{bmatrix} n \\ i \end{bmatrix}_q (-1)^i \frac{\left(q^{i-n}\Delta\right)^i \beta_{n-i}}{(\omega;\, q)_i} \qquad (i = 0, 1, \ldots, n).$$

□

Lemma 2.7 (Generalized de Casteljau algorithm)**.** *Given the polynomial* (2.6)*, let the quantities $\beta_i^{(k)}$ $(k = 0, 1, \ldots, n;\ i = 0, 1, \ldots, n-k)$ be defined in the following recursive way:*

$$\beta_i^{(0)} := \beta_i \qquad (i = 0, 1, \ldots, n); \tag{2.10}$$

$$\beta_i^{(k)} := \frac{1 - xq^{n-k-i}}{1-\omega q^{n-k}}\,\beta_i^{(k-1)} + \left(1 - \frac{1-xq^{n-k-i}}{1-\omega q^{n-k}}\right)\beta_{i+1}^{(k-1)} \qquad (k = 1, 2, \ldots, n;\ i = 0, 1, \ldots, n-k). \tag{2.11}$$

Then $p(x) = \beta_0^{(n)}$.

Proof. By using part *(iii)* of Lemma 2.1, we obtain

$$\begin{aligned} p(x) &= \sum_{i=0}^{n} \beta_i \, B_i^n(x;\omega|\, q) \\ &= \sum_{i=0}^{n} \beta_i \left(\frac{1-xq^{n-i-1}}{1-\omega q^{n-1}} \, B_i^{n-1}(x;\omega|\, q) + q^{n-i} \, \frac{x-\omega q^{i-1}}{1-\omega q^{n-1}} \, B_{i-1}^{n-1}(x;\omega|\, q) \right) \\ &= \sum_{i=0}^{n-1} \beta_i^{(1)} \, B_i^{n-1}(x;\omega|\, q), \end{aligned}$$

$\beta_i^{(1)}$ being defined according (2.11). Repeating the above process n times we arrive in

$$p(x) = \sum_{i=0}^{n-k} \beta_i^{(k)} \, B_i^{n-k}(x;\omega|\, q) \qquad (k = 1, 2, \ldots, n).$$

The last form is $p(x) = \beta_0^{(n)}$. □

Lemma 2.8 (Degree elevation)**.** *The nth degree polynomial* (2.6) *can be represented in the generalized Bernstein basis of degree* $n+1$*,*

$$p(x) = \sum_{i=0}^{n+1} \beta_i^* \, B_i^{n+1}(x;\omega|\, q), \tag{2.12}$$

where

$$\beta_i^* := \frac{[\, n-i+1\,]_q}{[\, n+1\,]_q} \beta_i + \left(1 - \frac{[\, n-i+1\,]_q}{[\, n+1\,]_q}\right) \beta_{i-1} \qquad (i = 0, 1, \ldots, n+1). \tag{2.13}$$

Proof. It suffice to use formula of part *(v)* of Lemma 2.1 in the expression (2.6) for the polynomial p. □

The next two lemmas give a representation of the polynomials (1.1) in terms of the *q-Pochhammer polynomials* $(x;\, q)_k$, as well as the so-called *inverse representation.*

Lemma 2.9. *For $n \in \mathbb{N}$ and $0 \le i \le n$, the following relation holds:*

$$B_i^n(x;\omega|\, q) = (-1)^i \, q^{\frac{1}{2}i(i+1)-in} \begin{bmatrix} n \\ i \end{bmatrix}_q \sum_{k=0}^{i} \begin{bmatrix} i \\ k \end{bmatrix}_q (-1)^k \, q^{\binom{k}{2}} \, \frac{(x;\, q)_{n-k}}{(\omega;\, q)_{n-k}}. \tag{2.14}$$

Proof. First observe that for any $n \in \mathbb{N}$ and $i = 0$, it takes the form

$$B_0^n(x;\omega|\, q) = \frac{(x;\, q)_n}{(\omega;\, q)_n},$$

which is in agreement with the definition (1.1). In the remaining part of the proof we use induction on n. Write formula (2.14) in the form

$$B_i^n(x;\omega|\, q) = \sum_{k=0}^{i} \alpha_{i,k}^{(n)} \, (x;\, q)_{n-k},$$

where

$$\alpha_{i,k}^{(n)} := (-1)^{i+k} q^{\binom{i+1}{2}+\binom{k}{2}-in} \begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} i \\ k \end{bmatrix}_q \frac{1}{(\omega;\, q)_{n-k}},$$

and assume that it holds for a certain n and for $0 \le i \le n$. By using (2.1), we obtain for $i = 1, 2, \ldots, n+1$

$$\begin{aligned} B_i^{n+1}(x;\omega|\, q) &= C\,(\omega q^{i-1} - x)\, B_{i-1}^{n}(x;\omega|\, q) \\ &= C\,(\omega q^{i-1} - x) \sum_{k=0}^{i-1} \alpha_{i-1,k}^{(n)}\,(x;\, q)_{n-k} \\ &= C \sum_{k=0}^{i} q^{k-n} \{(1 - xq^{n-k}) - (1 - \omega q^{n-k+i-1})\} \alpha_{i-1,k}^{(n)}\,(x;\, q)_{n-k} \\ &= C\, q^{-n} \sum_{k=0}^{i} q^{k} \alpha_{i-1,k}^{(n)} \big\{\, (x;\, q)_{n+1-k} - (1 - \omega q^{n-k+i-1})\,(x;\, q)_{n-k} \big\} \\ &= C\, q^{-n} \sum_{k=0}^{i} q^{k} \big\{ \alpha_{i-1,k}^{(n)} - q^{-1}(1 - \omega q^{n-k+i}) \alpha_{i-1,k-1}^{(n)} \big\}\,(x;\, q)_{n+1-k} \\ &= \sum_{k=0}^{i} \alpha_{i,k}^{(n+1)}\,(x;\, q)_{n+1-k}, \end{aligned}$$

where $C := [\,n+1\,]_q / ([\,i\,]_q(\omega q^n - 1))$. (We adopted the convention that $\alpha_{i-1,i}^{(n)} = \alpha_{i-1,-1}^{(n)} = 0$.) □

Lemma 2.10. *The Pochhammer polynomials have the following representation in the generalized Bernstein polynomial basis:*

$$\text{(2.15)} \qquad \frac{(x;\, q)_j}{(\omega;\, q)_j} = \begin{bmatrix} n \\ j \end{bmatrix}_q^{-1} \sum_{m=0}^{n-j} q^{jm} \begin{bmatrix} n-m \\ j \end{bmatrix}_q B_m^n(x;\omega|\, q) \qquad (0 \le j \le n;\;\; n \in \mathbb{N}).$$

Proof. First observe that for any $n \in \mathbb{N}$ and $j = n$ it reduces to

$$\frac{(x;\, q)_n}{(\omega;\, q)_n} = B_0^n(x;\omega|\, q),$$

hence is obviously true. In the remaining part of the proof we use induction on n. Obviously, (2.15) is true for $n = j = 0$. Assume that it holds for a certain n and $0 \le j \le n$, then use

Lemma 2.8 to obtain

$$\begin{aligned}
\frac{(x;\,q)_j}{(\omega;\,q)_j} &= \sum_{m=0}^{n-j} \beta_{j,m}^{(n)}\, B_m^n(x;\omega|\,q) \\
&= \sum_{m=0}^{n+1-j} \Bigg(\frac{[\,n+1-m\,]_q}{[\,n+1\,]_q}\, \beta_{j,m}^{(n)} \\
&\qquad\qquad + \left(1 - \frac{[\,n+1-m\,]_q}{[\,n+1\,]_q}\right) \beta_{j,m-1}^{(n)} \Bigg)\, B_m^{n+1}(x;\omega|\,q) \\
&= \sum_{m=0}^{n+1-j} \beta_{j,m}^{(n+1)}\, B_m^{n+1}(x;\omega|\,q) \qquad (0 \le j \le n),
\end{aligned}$$

where

$$\beta_{j,m}^{(n)} := q^{jm} \begin{bmatrix} n \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} n-m \\ j \end{bmatrix}_q .$$

Thus, identity (2.15) holds for any $n \in \mathbb{N}$ and $0 \le j \le n$. □

3. Big q–Jacobi polynomial expansion of generalized Bernstein polynomials.

Recall that the *big q-Jacobi polynomials* are defined by (see, e.g., [6, (7.3.10)], or [7, § 3.5])

$$P_k(x;a,b,c|q) := {}_3\phi_2\left(\begin{matrix} q^{-k}, abq^{k+1}, x \\ aq, cq \end{matrix} \,\middle|\, q;q\right) \qquad (k \ge 0), \tag{3.1}$$

and the *q-Hahn polynomials* are given by (see, e.g., [6, Eq. (7.3.21)], or [7, § 3.6])

$$Q_k(q^{-x};a,b,N|q) := {}_3\phi_2\left(\begin{matrix} q^{-k}, abq^{k+1}, q^{-x} \\ aq, q^{-N} \end{matrix} \,\middle|\, q;q\right) \quad (k = 0,1,\ldots,N;\ N \in \mathbb{N}). \tag{3.2}$$

We will prove the following formulas relating the generalized Bernstein, big q-Jacobi and q-Hahn polynomials.

Theorem 3.1. *Generalized Bernstein polynomials have the following representation in the big q-Jacobi polynomial basis:*

$$\begin{aligned}
B_i^n(x;\omega|\,q) =& (-1)^i q^{\binom{n-i}{2}}(-aq)^n \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{(bq;\,q)_n}{(abq^2;\,q)_n} \\
&\times \sum_{j=0}^{n} q^{-\binom{j+1}{2}} \left(-\frac{q^n}{a}\right)^j \frac{(abq^2;\,q)_{2j}\,(aq,abq,q^{-n};\,q)_j}{(abq;\,q)_{2j}\,(q,bq,abq^{n+2};\,q)_j} \\
&\times Q_{n-j}\left(q^{i-n};\frac{q^{-n-1}}{b},\frac{q^{-n-1}}{a},n\right|q\Big)\,P_j\left(x;a,b,\frac{\omega}{q}\right|q\Big); \qquad (3.3)
\end{aligned}$$

$$\begin{aligned}
B_i^n(x;\omega|\,q) =& (-1)^i q^{\binom{n-i}{2}}(-cq)^n \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{(b\omega/c;\,q)_n}{(bq\omega;\,q)_n} \\
&\times \sum_{j=0}^{n} q^{-\binom{j+1}{2}} \left(-\frac{q^n}{c}\right)^j \frac{(bq\omega;\,q)_{2j}\,(cq,b\omega,q^{-n};\,q)_j}{(b\omega;\,q)_{2j}\,(q,b\omega/c,b\omega q^{n+1};\,q)_j} \\
&\times Q_{n-j}\left(q^{i-n};\frac{cq^{-n}}{b\omega},\frac{q^{-n-1}}{c},n\right|q\Big)\,P_j\left(x;\frac{\omega}{q},b,c\right|q\Big). \qquad (3.4)
\end{aligned}$$

Proof. By inserting (cf. [8])

$$(x;\,q)_k = \frac{(aq,cq;\,q)_k}{(abq^2;\,q)_k}\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}}\begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(abq^2;\,q)_{2j}\,(abq;\,q)_j}{(abq;\,q)_{2j}\,(abq^{k+2};\,q)_j}\,P_j(x;a,b,c|q)$$

into (2.14), we obtain

$$B_i^n(x;\omega|\,q) = (-1)^i\,q^{\frac{1}{2}i(i+1)-in}\begin{bmatrix} n \\ i \end{bmatrix}_q \sum_{j=0}^{n} C_j(\omega)\,P_j(x;a,b,c|q),$$

where

$$\begin{aligned}
C_j(\omega) =& (-1)^j q^{\binom{j}{2}}\frac{(abq^2;\,q)_{2j}\,(abq;\,q)_j}{(abq;\,q)_{2j}} \\
&\times \sum_{k=0}^{\min(i,n-j)}\begin{bmatrix} n-k \\ j \end{bmatrix}_q \begin{bmatrix} i \\ k \end{bmatrix}_q \frac{(-1)^k q^{\binom{k}{2}}\,(aq,cq;\,q)_{n-k}}{(\omega,abq^2;\,q)_{n-k}\,(abq^{n-k+2};\,q)_j}.
\end{aligned}$$

Using properties of the q-Pochhammer symbol (see, e.g., [6], or [7, § 0.2]), we obtain

$$\begin{aligned}
C_j(\omega) =& \frac{(aq;\,q)_n\,(cq;\,q)_n}{(abq^2;\,q)_n\,(\omega;\,q)_n}(-1)^j q^{\binom{j}{2}}\frac{(abq^2;\,q)_{2j}\,(abq,q^{n+1-j};\,q)_j}{(abq;\,q)_{2j}\,(q,abq^{n+2};\,q)_j} \\
&\times \sum_{k=0}^{\min(i,n-j)} \frac{(q^{j-n},q^{-i},q^{1-n}/\omega,q^{-n-j-1}/(ab);\,q)_k}{(q,q^{-n},q^{-n}/a,q^{-n}/c;\,q)_k}\left(\frac{b\,\omega}{c}\,q^i\right)^k.
\end{aligned}$$

Hence, we have the formula

$$B_i^n(x;\omega|\,q) = (-1)^i q^{\binom{i+1}{2}-in} \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{(aq, cq;\, q)_n}{(abq^2, \omega;\, q)_n} \sum_{j=0}^{n} q^{jn} \frac{(abq^2;\, q)_{2j}\,(abq, q^{-n};\, q)_j}{(abq;\, q)_{2j}\,(q, abq^{n+2};\, q)_j} \times {}_4\phi_3\left(\begin{matrix} q^{-i},\, q^{j-n},\, q^{1-n}/\omega,\, q^{-n-j-1}/(ab) \\ q^{-n}, q^{-n}/a,\, q^{-n}/c \end{matrix}\,\middle|\, q; \frac{\omega b q^i}{c}\right) P_j(x;a,b,c|q). \tag{3.5}$$

By setting $c = \omega/q$ in the above formula, and applying the identity (cf. [1, Eq. (10.10.5)])

$$\begin{aligned} &{}_3\phi_2\left(\begin{matrix} q^{j-n},\, q^{-i},\, q^{-n-j-1}/(ab) \\ q^{-n},\, q^{-n}/a \end{matrix}\,\middle|\, q; bq^{i+1}\right) \\ &= q^{(n-j)(j+n+1)}\,(ab)^{n-j} \frac{(q^{-n}/b;\, q)_{n-j}}{(aq^{j+1};\, q)_{n-j}}\, {}_3\phi_2\left(\begin{matrix} q^{j-n},\, q^{i-n},\, q^{-n-j-1}/(ab) \\ q^{-n},\, q^{-n}/b \end{matrix}\,\middle|\, q; q\right), \end{aligned}$$

we obtain after some algebra

$$\begin{aligned} B_i^n(x;\omega|\,q) &= (-1)^i q^{\frac12 i(i+1)-in} \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{(bq;\, q)_n}{(abq^2;\, q)_n} \\ &\quad\times \sum_{j=0}^{n} q^{\frac12(n-j)(j+n+1)+jn}(-a)^{n-j} \frac{(abq^2;\, q)_{2j}\,(aq, abq, q^{-n};\, q)_j}{(abq;\, q)_{2j}\,(q, bq, abq^{n+2};\, q)_j} \\ &\quad\quad\times {}_3\phi_2\left(\begin{matrix} q^{j-n},\, q^{i-n},\, q^{-n-j-1}/(ab) \\ q^{-n}, q^{-n}/b \end{matrix}\,\middle|\, q; q\right) P_j(x;a,b,\omega/q|q) \\ &= (-1)^{n-i} q^{\frac12 n(n+1)+\frac12 i(i+1)-in} a^n \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{(bq;\, q)_n}{(abq^2;\, q)_n} \\ &\quad\times \sum_{j=0}^{n} q^{jn-\frac12 j(j+1)}(-a)^{-j} \frac{(abq^2;\, q)_{2j}\,(aq, qbq, q^{-n};\, q)_j}{(abq;\, q)_{2j}\,(q, bq, abq^{n+2};\, q)_j} \\ &\quad\quad\times Q_{n-j}(q^{i-n}; q^{-n-1}/b, q^{-n-1}/a, n|q)\, P_j(x;a,b,\omega/q|q), \end{aligned}$$

which completes the proof of (3.3).

Formula (3.4) follows in a similar way, by setting $a = \omega/q$ in (3.5), and applying again [1, Eq. (10.10.5)]. □

Remember that *little q-Jacobi polynomials* are given by (see, e.g., [7, § 3.12])

$$p_k(x;\alpha,\beta|q) := {}_2\phi_1\left(\begin{matrix} q^{-k},\, \alpha\beta q^{k+1} \\ \alpha q \end{matrix}\,\middle|\, q; qx\right) \qquad (k \ge 0). \tag{3.6}$$

Corollary 3.2. *q-Bernstein polynomials* (1.6) *have the following representation in the little q-Jacobi polynomial basis:*

$$b_i^n(x;q) = (-1)^{n-i} q^{\binom{n-i}{2}} a^n \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{(bq;\,q)_n}{(abq^2;\,q)_n} \times \sum_{j=0}^{n} q^{n(j+1)} \frac{(abq^2;\,q)_{2j}\,(abq, q^{-n};\,q)_j}{(abq;\,q)_{2j}\,(q, abq^{n+2};\,q)_j} \times Q_{n-j}\left(q^{i-n}; \frac{1}{bq^{n+1}}, \frac{1}{aq^{n+1}}, n \middle|\, q\right) p_j\left(\frac{x}{aq}; b, a \middle|\, q\right). \tag{3.7}$$

Notice that equation (3.7) is equivalent to a formula obtained in [2].

Proof. By setting $\omega = 0$ in (3.3), using (1.8), and the relation [7, § 3.5]

$$P_j(x;a,b,0|\,q) = \frac{(bq;\,q)_j}{(aq;\,q)_j}(-aq)^j q^{\binom{j}{2}} p_j\left(\frac{x}{aq}; b, a \middle|\, q\right) \tag{3.8}$$

the result follows. □

Recall that the *Jacobi polynomials* are defined by (see, e.g., [1, p. 99], or [7, § 1.8])

$$P_k^{(\alpha,\beta)}(x) := \frac{(\alpha+1)_k}{k!}\, {}_2F_1\left(\begin{matrix} -k, k+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \middle|\, \frac{1-x}{2}\right) \qquad (k \geq 0), \tag{3.9}$$

while the *Hahn polynomials* are given by [7, § 1.5]

$$Q_k(x;\alpha,\beta,N) := {}_3F_2\left(\begin{matrix} -k, k+\alpha+\beta+1, -x \\ \alpha+1, -N \end{matrix} \middle|\, 1\right) \quad (k = 0,1,\ldots,N;\ N \in \mathbb{N}). \tag{3.10}$$

Corollary 3.3. *Bernstein polynomials* (1.4) *have the following representation in the Jacobi polynomial basis* (3.9)*:*

$$B_i^n(x) = (-1)^{n-i}\binom{n}{i}\frac{(\beta+1)_n}{(\alpha+\beta+2)_n} \times \sum_{j=0}^{n}(-1)^j \frac{(\alpha+\beta+2)_{2j}(\alpha+\beta+1,-n)_j}{(\alpha+\beta+1)_{2j}(\beta+1,\alpha+\beta+n+2)_j} \times Q_{n-j}(n-i; -\beta-n-1, -\alpha-n-1, n)\, P_j^{(\alpha,\beta)}(2x-1). \tag{3.11}$$

Notice that equation (3.11) is equivalent to a formula obtained in [13].

Proof. The result follows by setting $a = q^\alpha$, $b = q^\beta$ in (3.7), letting $q \uparrow 1$, using (1.7) and (cf. [7, § 5.6])

$$\lim_{q\uparrow 1} p_j(x; q^\beta, q^\alpha |\, q) = (-1)^j \frac{j!}{(\beta+1)_j} P_j^{(\alpha,\beta)}(2x-1), \tag{3.12}$$

$$\lim_{q\uparrow 1} Q_{n-j}\left(q^{i-n}; q^{-\beta-n-1}, q^{-\alpha-n-1}, n \middle|\, q\right) = Q_{n-j}(n-i; -\beta-n-1, -\alpha-n-1, n)\,. \tag{3.13}$$

□

Corollary 3.4. *Discrete Bernstein polynomials* (1.5) *can be represented in the form*

$$b_i^n(N,t) = (-1)^i \binom{n}{i} \frac{(\beta+1)_n}{(\alpha+\beta+2)_n} \times \sum_{j=0}^{n} (-1)^j \frac{(\alpha+\beta+2)_{2j}(\alpha+1, \alpha+\beta+1, -n)_j}{(\alpha+\beta+1)_{2j}(1, \beta+1, \alpha+\beta+n+2)_j} \times Q_{n-j}(i; -\beta-n-1, -\alpha-n-1, n)\, Q_j(t; \alpha, \beta, N)\,. \tag{3.14}$$

Note that a formula equivalent to (3.14) has been obtained in [13].

Proof. The result follows from (3.3) by setting $a = q^\alpha$, $b = q^\beta$, $\omega = q^{-N}$, $x = q^{-t}$, letting $q \uparrow 1$, and using (1.8), (3.13) and (cf. [7, § 5.6])

$$\lim_{q\uparrow 1} P_j(q^{-t}; q^\alpha, q^\beta, q^{-N-1}|\, q) = \lim_{q\uparrow 1} Q_j(q^{-t}; q^\alpha, q^\beta, N|\, q) = Q_j(t; \alpha, \beta, N). \tag{3.15}$$

□

4. Generalized Bernstein polynomial expansion of big q–Jacobi polynomials.

Let us recall that the *dual q-Hahn polynomials* are defined by [7, § 3.7]

$$R_k\left(\mu(x); \gamma, \delta, N|q\right) := {}_3\phi_2\left(\begin{matrix} q^{-k}, q^{-x}, \gamma\delta q^{x+1} \\ \gamma q, q^{-N} \end{matrix} \middle| q; q\right)$$

$$(\mu(x) := q^{-x} + \gamma\delta q^{x+1};\ k = 0, 1, \ldots, N;\ N \in \mathbb{N}).$$

We prove the following.

Theorem 4.1. *Big q-Jacobi polynomials* (3.1) *have the following representation in the generalized Bernstein polynomial basis:*

$$P_i(x; a, b, c|q) = \sum_{j=0}^{n} R_{n-j}\left(q^{-i} + abq^{i+1}; a, b, n|q\right)\, B_j^n(x; cq|\, q). \tag{4.1}$$

Proof. By inserting (2.15) into the expansion [7, § 3.5]

$$P_i(x; a, b, c|q) = \sum_{k=0}^{i} \frac{(q^{-i}, abq^{i+1}, x;\, q)_k}{(q, aq, cq;\, q)_k}\, q^k,$$

we obtain after some algebra

$$P_i(x; a, b, c|q) = \sum_{j=0}^{n} \left(\sum_{k=0}^{\min(i,n-j)} \frac{(q^{-i}, q^{j-n}, \omega, abq^{i+1};\, q)_k}{(q, aq, cq, q^{-n};\, q)_k}\, q^k \right) B_j^n(x; \omega|\, q).$$

Hence

$$P_i(x; a, b, c|q) = \sum_{j=0}^{n} {}_4\phi_3\left(\begin{matrix} q^{-i},\ q^{j-n},\ \omega,\ abq^{i+1} \\ q^{-n}, aq,\ cq \end{matrix} \middle| q; q\right) B_j^n(x; \omega|\, q).$$

Setting $\omega = cq$ in the above equation, we obtain

$$P_i(x;a,b,c|q) = \sum_{j=0}^{n} {}_3\phi_2\left(\begin{matrix} q^{-i},\, q^{j-n},\, abq^{i+1} \\ q^{-n}, aq \end{matrix}\,\middle|\, q;q\right) B_j^n(x;cq|\,q)$$
$$= \sum_{j=0}^{n} R_{n-j}\left(q^{-i}+abq^{i+1};a,b,n|q\right) B_j^n(x;cq|\,q).$$

□

Corollary 4.2. *Little q-Jacobi polynomials* (3.6) *and q-Bernstein polynomials* (1.6) *are connected by the formula*

$$p_i\left(x;a,b\middle|\,q\right) = (-b)^{-i}q^{-\binom{i+1}{2}}\frac{(bq;\,q)_i}{(aq;\,q)_i} \times \sum_{j=0}^{n} R_{n-j}\left(q^{-i}+abq^{i+1};b,a,n|q\right) b_j^n(bqx;q). \tag{4.2}$$

Remark that a formula equivalent to (4.2) has been recently obtained in [2].

Proof. The result follows by setting $c = 0$ in (4.1), using (3.8), replacing x by aqx, and interchanging the roles of the parameters a and b. □

The *dual Hahn polynomials* are defined by [7, § 1.6]

$$R_k\left(\lambda(x);\gamma,\delta,N\right) := {}_3F_2\left(\begin{matrix} -k,-x,x+\gamma+\delta+1 \\ \gamma+1,-N \end{matrix}\,\middle|\,1\right)$$
$$(\lambda(x) := x(x+\gamma+\delta+1);\;\; k=0,1,\ldots,N;\;\; N\in\mathbb{N}).$$

Corollary 4.3. *Jacobi polynomials* (3.9) *and Bernstein polynomials* (1.4) *are connected by the formula*

$$P_i^{(\alpha,\beta)}(2x-1) = \frac{(\alpha+1)_i}{i!}\sum_{j=0}^{n} R_{n-j}\left(i(i+\alpha+\beta+1);\alpha,\beta,n\right) B_j^n(x). \tag{4.3}$$

Remark that a formula equivalent to (4.3) was obtained in [4] .

Proof. The argument is similar to the one used in the proof of Cor. 3.3. □

Corollary 4.4. *The Hahn polynomials* (3.10) *have the following representation in the discrete Bernstein polynomial basis* (1.5)*:*

$$Q_i\left(x;\alpha,\beta,N\right) = \sum_{j=0}^{n} R_j\left(i(i+\alpha+\beta+1);\alpha,\beta,n\right) b_j^n(N,x). \tag{4.4}$$

Note that Eq. (4.4) has been earlier obtained in [15].

Proof. The argument is similar to the one used in the proof of Cor. 3.4. □

References

[1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge Univ. Press, Cambridge, 1999.

[2] I. Area, E. Godoy, P. Woźny, S. Lewanowicz, and A. Ronveaux, *Formulae relating q–Bernstein, little q–Jacobi and q–Hahn polynomials: Application to the q-Bézier curve evaluation*, Integr. Transf. Spec. Funct., 15 (2004), 375–385.

[3] E. W. Cheney, *Introduction to Approximation Theory*, second ed., Chelsea Publ., Providence, R. I., 1982.

[4] Z. Ciesielski, *Explicit formula relating the Jacobi, Hahn and Bernstein polynomials*, SIAM J. Math. Anal., 18 (1987), pp. 1573–1575.

[5] G. E. Farin, *Curves and Surfaces for Computer–Aided Geometric Design. A Practical Guide*, third ed., Academic Press, Boston, 1996.

[6] G. Gasper, and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications, Vol. 35, Cambridge University Press, Cambridge, 1990.

[7] R. Koekoek and R. F. Swarttouw, *The Askey scheme of hypergeometric orthogonal polynomials and its q-analogue*, Fac. Techn. Math. Informatics, Delft Univ. of Technology, Rep. 98–17, Delft, 1998.

[8] S. Lewanowicz, E. Godoy, I. Area, A. Ronveaux, and A. Zarzo, *Recurrence relations for the coefficients of the Fourier series expansions with respect to q-classical orthogonal polynomials*, Numer. Algorithms, 23 (2000), pp. 31–50.

[9] S. Lewanowicz, P. Woźny, *Generalized Bernstein polynomials*, BIT Numer. Math. 44 (2004), 63–78.

[10] H. Oruç, and G. M. Philips, *q-Bernstein polynomials and Bézier curves*, J. Comput. Appl. Math., 151 (2003), pp. 1–12.

[11] G. M. Phillips, *A de Casteljau algorithm for generalized Bernstein polynomials*, BIT, 36 (1996), pp. 232–236.

[12] G. M. Phillips, *Bernstein polynomials based on the q-integers*, Ann. Numer. Math., 4 (1997), pp. 511–518.

[13] A. Ronveaux, A. Zarzo, I. Area, and E. Godoy, *Bernstein bases and Hahn–Eberlein orthogonal polynomials*, Integr. Transf. Spec. Funct., 7 (1998), pp. 87–96.

[14] P. Sablonnière, *Discrete Bézier curves and surfaces*, in Mathematical Methods in Computer Aided Geometric Design II, T. Lyche and L. L. Schumaker, eds., Academic Press, New York, 1992, pp. 497–515.

[15] P. Sablonnière, *Discrete Bernstein bases and Hahn polynomials*, J. Comput. Appl. Math., 49 (1993), pp. 233–241.

Institute of Computer Science, University of Wrocław, Poland
E-mail address: {Stanislaw.Lewanowicz,Pawel.Wozny}@ii.uni.wroc.pl