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Two-variable orthogonal polynomials of big q-Jacobi type

Stanisław Lewanowicz, Paweł Woźny

Institute of Computer Science, University of Wrocław, ul. F. Joliot-Curie 15, 50-383 Wrocław, Poland

Abstract

A four-parameter family of orthogonal polynomials in two variables is defined by

$$P_{n,k}(x,y;a,b,c,d;q) := P_{n-k}(y;a,bcq^{2k+1},dq^k;q)\,y^k(dq/y;\,q)_k\,P_k\,(x/y;c,b,d/y;q) \qquad (n\in\mathbb{N};\ k=0,1,\ldots,n),$$

where $q\in(0,\,1)$, $0<aq,\,bq,\,cq<1$, $d<0$, and $P_m(t;\alpha,\beta,\gamma;q)$ are univariate big q-Jacobi polynomials,

$$P_m(t;\alpha,\beta,\gamma;q) := {}_3\phi_2\left(\begin{matrix} q^{-m},\alpha\beta q^{m+1},t \\ \alpha q,\gamma q\end{matrix}\,\middle|\, q;q\right) \qquad (m\ge 0)$$

(see, e.g., [1, Section 7.3]). These polynomials form an extension of Dunkl's bivariate (little) q-Jacobi polynomials [2]. We prove orthogonality property of the new polynomials, and show that they satisfy a three-term relation in a vector-matrix notation, as well as a second-order partial q-difference equation.

We give some basic properties of the new polynomials. First, we show that they form an orthogonal system with respect to the linear functional $\boldsymbol{u}$ defined by

$$\langle\boldsymbol{u},p\rangle := \int_{dq}^{aq}\int_{dq}^{cqy}\frac{(dq/y,x/(cy),x/d,y/a,y/d;\,q)_\infty}{y\,(d/(cy),cqy/d,x/y,bx/d,y;\,q)_\infty}\,p(x,y)\,\mathrm{d}_qx\,\mathrm{d}_qy,$$

where p is any two-variable polynomial. Second, we proved that the following three-term relations hold:

$$\left.\begin{aligned} x\,\mathbb{P}_n &= A_{n,1}\mathbb{P}_{n+1}+B_{n,1}\mathbb{P}_n+C_{n,1}\mathbb{P}_{n-1},\\ y\,\mathbb{P}_n &= A_{n,2}\mathbb{P}_{n+1}+B_{n,2}\mathbb{P}_n+C_{n,2}\mathbb{P}_{n-1}\end{aligned}\right\}\quad (n\ge 0),$$

where we used the notation $\mathbb{P}_n := [P_{n,0},\,P_{n,1},\,\ldots,P_{n,n}]^T$, and $A_{n,i}$, $B_{n,i}$ and $C_{n,i}$ are tridiagonal ($i=1$) or diagonal ($i=2$) matrices of appropriate dimensions. Last, we showed that for any $n\ge 0$, the polynomial vector $\mathbb{P}_n$ satisfies the linear second-order partial q-difference equation in the form

$$\begin{aligned} l_{11}\,D_{q,x}D_{q^{-1},x}\mathbb{P}_n + l_{22}D_{q,y}D_{q^{-1},y}\mathbb{P}_n + l_{12}^{-}D_{q^{-1},x}D_{q^{-1},y}\mathbb{P}_n&\\ + l_{12}^{+}D_{q,x}D_{q,y}\mathbb{P}_n + m_1D_{q,x}\mathbb{P}_n + m_2D_{q,y}\mathbb{P}_n &= \lambda_n\,\mathbb{P}_n,\end{aligned}$$

with $\lambda_n := [\,n\,]_q q^{1-n}(abcq^{n+2}-1)/(q-1)$, and the polynomial coefficients $l_{11}(x)$, $l_{22}(y)$, $l_{12}^{\pm}(x,y)$, $m_1(x)$, $m_2(y)$ being given explicitly. Here $D_{q^{\pm1},x}$ and $D_{q^{\pm1},y}$ are partial q-derivative operators, and we let

$$D_{q^{\pm1},z}\mathbb{P}_n := [D_{q^{\pm1},z}P_{n,0},\,D_{q^{\pm1},z}P_{n,1},\,\ldots,D_{q^{\pm1},z}P_{n,n}]^T \qquad (z=x,\,y).$$

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