Two-variable orthogonal polynomials of big \( q \)-Jacobi type

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Abstract

A four-parameter family of orthogonal polynomials in two variables is defined by

\[
P_{n,k}(x,y; a,b,c,d; q) := P_{n-k}(y; a, bq^{k+1}, dq^k; q) y^k (dq/y; q)_k P_k (x/y; c, b, d; y; q)
\]

\((n \in \mathbb{N}; k = 0,1, \ldots , n)\),

where \( q \in (0,1) \), \( 0 < aq, bq, cq < 1, d < 0 \), and \( P_m(t; \alpha, \beta, \gamma; q) \) are univariate \( q \)-Jacobi polynomials,

\[
P_m(t; \alpha, \beta, \gamma; q) := \frac{\phi_m(t)}{\phi_0(t)} \quad (m \geq 0)
\]

(see, e.g., [1, Section 7.3]). These polynomials form an extension of Dunkl’s bivariate (little) \( q \)-Jacobi polynomials [2]. We prove orthogonality property of the new polynomials, and show that they satisfy a three-term relation in a vector-matrix notation, as well as a second-order partial \( q \)-difference equation.

We give some basic properties of the new polynomials. First, we show that they form an orthogonal system with respect to the linear functional \( u \) defined by

\[
(u, p) := \int_{a}^{b} \int_{d}^{c} (dq/y, x/(cy), x/d, y/a, y/d; q)_\infty p(x,y) dx \, dq,
\]

where \( p \) is any two-variable polynomial. Second, we proved that the following three-term relations hold:

\[
x P_n = A_{n,1} P_{n+1} + B_{n,1} P_n + C_{n,1} P_{n-1},
\]

\[
y P_n = A_{n,2} P_{n+1} + B_{n,2} P_n + C_{n,2} P_{n-1}
\]

\((n \geq 0)\),

where we used the notation \( P_n := [P_{n,0}, P_{n,1}, \ldots , P_{n,n}]^T \), and \( A_{n,i}, B_{n,i} \) and \( C_{n,i} \) are tridiagonal \((i = 1)\) or diagonal \((i = 2)\) matrices of appropriate dimensions. Last, we showed that for any \( n \geq 0 \), the polynomial vector \( P_n \) satisfies the linear second-order partial \( q \)-difference equation in the form

\[
l_{11} D_{q,z} x D_{q^{-1},z} P_n + l_{22} D_{q,y} D_{q^{-1},y} P_n + l_{12} D_{q^{-1},x} D_{q,y} P_n
\]

\[
+ l_{12} D_{q,x} D_{q,y} P_n + m_1 D_{q,x} P_n + m_2 D_{q,y} P_n = \lambda_n P_n,
\]

with \( \lambda_n := [n]_q q^{1-n}(abcq^{n+2}-1)/(q-1) \), and the polynomial coefficients \( l_{ij}(x), l_{ij}(y), l_{ij}^1(x, y), m_1(x), m_2(y) \) being given explicitly. Here \( D_{q,z}, D_{q,y} \) are partial \( q \)-derivative operators, and we let

\[
D_{q,z} x P_n := [D_{q,z} x P_{n,0}, D_{q,z} x P_{n,1}, \ldots , D_{q,z} x P_{n,n}]^T \quad (z = x, y).
\]

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