Regular quasi-nonexpansive operators and their applications

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Abstract

An operator T defined on a Hilbert space $\mathcal{H}$ is called quasi-nonexpansive if $\|Tx-z\| \leq \|x-z\|$ for any $x \in \mathcal{H}$ and any $z \in \operatorname{Fix} T$. An example of such an operator is a nonexpansive one having a fixed point, e.g. the metric projection onto a nonempty closed convex subset. Quasi-nonexpansive operators play an important role in iterative processes for solving fixed point problems, convex feasibility problems, minimization problems, variational inequalities etc. These problems are mathematical models for many practical problems which arise in physical, medical, technical and information sciences, e.g., in signal processing, image restoration, medical imaging, learning process etc. In the literature one can find many methods for solving these problems, e.g., the Kaczmarz method, the Cimmino method, the Landweber method, the Douglas–Rachford method, block iterative method, string averaging method, gradient projection method, hybrid steepest descent method etc. The weak convergence of these methods which apply nonexpansive operators having a fixed point follows from well known results of Opial and of Krasnosel'skiĭ and Mann. In general, the convergence of iterative processes which apply quasi-nonexpansive operators depends on the regularity of these operators. Roughly spoken, the regularity of an operator T defined on a finite dimensional space means that the displacement $Tx - x$ can be small only in a small neighborhood of the subset of fixed points of T. We present a systematic study of regular quasi-nonexpansive operators and of sequences of such operators in a Hilbert space. We are interested, in particular, in weakly, boundedly and linearly regular operators and sequences of operators. We give several examples of such operators. Further, we present the conditions under which these types of regularities is preserved under relaxations, convex combinations and composition of operators. Moreover, in this connection, we show that the weak regularity, regularity and linear regularity leads to weak, strong and linear convergence, respectively, of various iterative methods.