Theorem Proving (8): Well-Orders, Reduction Orders, Knuth-Bendix Completion

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1. Use Knuth-Bendix completion to show that
   (a) \( a \approx b, b \approx c, c \approx d \models a \approx d \),
   (b) \( x+y \approx y+x, x+s(y) \approx s(x+y), s(y)+x \models x+s(y) \approx s(y)+x \),
   (c) \( a \approx s(s(a)), s(s(a)) \approx s(t(a)), t(a) \approx b, t(s(b)) \models c \models s(c) \approx a \).

2. Do the previous task another time, but use a different order on the signature to start with.

3. Let \( \succ \) be the following order: (The real Knuth-Bendix order)
   (a) If \( \#t_1 > \#t_2 \), then \( t_1 \succ t_2 \).
   (b) If \( \#t_1 = \#t_2 \), then write \( t_1 \) in the form \( t_1 = f(\alpha_1, \ldots, \alpha_n) \), and write \( t_2 \) in the form \( t_2 = g(\beta_1, \ldots, \beta_m) \).
      i. If \( f \succ g \), then \( t_1 \succ t_2 \).
      ii. If \( f = g \), then let \( i \) be the smallest index for which \( \alpha_i \not\approx \beta_i \). If \( \alpha_i \succ \beta_i \), then \( t_1 \succ t_2 \).
   Show that \( \succ \) from the previous question is a reduction order.

4. Define \( + \) on ordinals by transfinite recursion. Define \( \times \) on ordinals by transfinite recursion.

5. Show that \( \beta \)-reduction is confluent.

6. Let the axiom of choice be the following: For every set \( S \) of sets there exists a function \( f \), s.t. for \( s \in S \) \( f(s) \in s \).
   Using axiom of choice, prove that every set \( S \) can be well-ordered. (There exists a wellfounded total order \( < \) on \( S \).