Theorem Proving: Exercises on Sequent Calculus
for Higher-Order Logic

13.03.2013

1. Prove the following formulas using the one-sided sequent calculus, that was
discussed today in class. First a proper context has to be constructed.
This context has to be typechecked. After that, the formulas can be
proven.
(a) \( \forall x \approx x \),
(b) \( \forall x s(x) \approx x \vdash \forall x s(s(x)) \approx x \).
(c) \( \forall x (P \rightarrow Q(x)) \vdash P \rightarrow \forall x Q(x) \).
(d) \( \forall x (P(x) \land Q) \vdash (\forall x P(x)) \land Q \).
(e) \( (\neg \forall x P(x)) \leftrightarrow \exists x \neg P(x) \).
(f) \( \exists x (D(x) \rightarrow \forall y D(y)) \), (the famous Drinker paradox.)
(g) \( \forall x f(x) \approx x, \forall x \exists y p(f(x), y) \vdash \forall x \exists y p(x, f(y)) \).

2. Consider the context

\[ D: \text{Set} \]
\[ R: \text{D} \rightarrow \text{D} \rightarrow \text{Prop} \]

Define the transitive, reflexive and symmetric closure of \( R \) as
(a) The smallest binary relation that is transitive, symmetric, reflexive,
and which includes \( R \).
(b) The smallest binary relation that is reflexive, and that is preserved
under chaining to the right with \( R \) or \( R^{-1} \).
(c) The smallest binary relation that is reflexive, and that is preserved
under chaining to the left with \( R \) or \( R^{-1} \).

3. Prove that the three definitions of the previous task are equivalent. The
proofs are essentially induction proofs.
An inductive set \( S \) is a set that is defined as the \( \subseteq \)-smallest set that satisfies
some set of rules.
If one wants to show that the elements in \( S \) have some property \( P \), it is
sufficient to show that \( P \) satisfies the same set of rules. By minimality of
\( S \), it follows that \( S \subseteq P \).
4. A relation $\prec$ is a well-order if it is an order (irreflexive and transitive), and in addition, it has the property that every non-empty set has minimal elements.

Well-orders can be used in transfinite induction.

If $\prec$ is a well-order, and $P$ is a property, s.t.

- for $d \in D$, whenever all $d' \prec d$ have property $P$, then also $d$ has property $P$,

then all $d \in D$ have property $P$.

(a) Formalize the well-order property.
(b) Formalize the transfinite induction principle.
(c) Prove correctness of the transfinite induction principle.