Curry-Howard Correspondence beyond Minimal Logic

Lecture Interactive Proof Tools

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Contents

- Minimal logic with conjunction and disjunction.

Minimal Logic + Conjunction/Disjunction

Additional natural deduction rules:

- Wedge introduction $$\frac{\Delta \vdash \varphi_1 \qquad \Delta \vdash \varphi_2}{\Delta \vdash \varphi_1 \wedge \varphi_2}$$
- Wedge eliminations $$\frac{\Delta \vdash \varphi_1 \wedge \varphi_2}{\Delta \vdash \varphi_1} \qquad \frac{\Delta \vdash \varphi_1 \wedge \varphi_2}{\Delta \vdash \varphi_2}$$
- Vee introductions $$\frac{\Delta \vdash \varphi_1}{\Delta \vdash \varphi_1 \vee \varphi_2} \qquad \frac{\Delta \vdash \varphi_2}{\Delta \vdash \varphi_1 \vee \varphi_2}$$
- Vee elimination $$\frac{\Delta \vdash \varphi_1 \vee \varphi_2 \qquad \Delta, \varphi_1 \vdash \psi \qquad \Delta, \varphi_2 \vdash \psi}{\Delta \vdash \psi}$$

Simply Typed λ-calculus + Pairs/Sums – Syntax

- Types Π: Formulas of minimal logic + conjunction/disjunction
- Contexts C: sets of variable/type assignments as before
- Pseudo-Terms Λ_Π: Defined by the grammar ($\varphi_1, \varphi_2 \in \Pi$)

$$\begin{aligned}
\Lambda_\Pi = {} & V \\
& \mid (\lambda V{:}\Pi.\Lambda_\Pi) \mid (\Lambda_\Pi\, \Lambda_\Pi) \\
& \mid \langle \Lambda_\Pi, \Lambda_\Pi \rangle \mid \pi_1(\Lambda_\Pi) \mid \pi_2(\Lambda_\Pi) \\
& \mid \mathrm{in}_1^{\varphi_1 \vee \varphi_2}(\Lambda_\Pi) \mid \mathrm{in}_2^{\varphi_1 \vee \varphi_2}(\Lambda_\Pi) \mid \mathrm{case}(\Lambda_\Pi, V.\Lambda_\Pi, V.\Lambda_\Pi)
\end{aligned}$$

Pseudo-Terms are defined modulo $\equiv_\alpha$. Note that the rules for $\equiv_\alpha$ must be extended to take care of the bound variables in case-terms.

Simply Typed λ-calculus + Pairs/Sums – Typing

Additional rules for the typability relation:

$$\frac{\Gamma \vdash M : \tau_1 \qquad \Gamma \vdash N : \tau_2}{\Gamma \vdash \langle M, N \rangle : \tau_1 \wedge \tau_2}$$

$$\frac{\Gamma \vdash M : \tau_1 \wedge \tau_2}{\Gamma \vdash \pi_1(M) : \tau_1} \qquad \frac{\Gamma \vdash M : \tau_1 \wedge \tau_2}{\Gamma \vdash \pi_2(M) : \tau_2}$$

$$\frac{\Gamma \vdash M : \tau_1}{\Gamma \vdash \mathrm{in}_1^{\tau_1 \vee \tau_2}(M) : \tau_1 \vee \tau_2} \qquad \frac{\Gamma \vdash M : \tau_2}{\Gamma \vdash \mathrm{in}_2^{\tau_1 \vee \tau_2}(M) : \tau_1 \vee \tau_2}$$

$$\frac{\Gamma \vdash M : \tau_1 \vee \tau_2 \qquad \Gamma, x : \tau_1 \vdash K : \sigma \qquad \Gamma, y : \tau_2 \vdash L : \sigma}{\Gamma \vdash \mathrm{case}(M, x.K, y.L) : \sigma}$$

Simply Typed λ-calculus + Pairs/Sums – Reduction

Extend β-reduction by the following *contractions*:

- $\pi_1(\langle M, N\rangle) \to_\beta M$
- $\pi_2(\langle M, N\rangle) \to_\beta N$
- $\mathrm{case}(\mathrm{in}_1^{\varphi_1 \vee \varphi_2}(M), x.K, y.L) \to_\beta K[x := M]$
- $\mathrm{case}(\mathrm{in}_2^{\varphi_1 \vee \varphi_2}(M), x.K, y.L) \to_\beta L[y := M]$

Note:

- The reduction properties (subject reduction, Church-Rosser, strong normalization) of $\to_\beta$ are preserved.
- The Curry-Howard correspondence continues to hold as for minimal logic (with conjunctions = pairs, disjunctions = sums).