# Completeness of Resolution

Lecture Interactive Proof Tools
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# Multiset Extension of an Ordering

- Let $\langle X, \succ \rangle$ be a partial order.
- For all multisets $M$ and $N$ over $X$ (i. e., functions $M, N : X \to \mathbb{N}$) define the **multiset extension** $\succ_{\text{mul}}$ of $\succ$ by $M \succ_{\text{mul}} N$ iff
  - $M \neq N$ and
  - $\forall x \in X : N(x) > M(x) \Rightarrow \exists y \in X : y \succ x \wedge M(y) > N(y)$.
- Examples: $[2] \succ_{\text{mul}} [1, 1]$, $[2, 2, 7, 3, 7] \succ_{\text{mul}} [6, 3, 6, 3, 6]$.

Properties of $\succ_{\text{mul}}$:

- $\succ_{\text{mul}}$ is a partial order.
- If $\succ$ well-founded then $\succ_{\text{mul}}$ well-founded.
- If $\succ$ total then $\succ_{\text{mul}}$ total.

# Clause Ordering

We define a total and well-founded ordering on clauses in stages.

- Choose total and well-founded ordering $\succ_{\text{var}}$ on prop. vars $R_0$.[a]
- For $A, B \in R_0$, define $\succ_{\text{lit}}$ on literals by
  - $A \succ_{\text{lit}} B$ if $A \succ_{\text{var}} B$,
  - $A \succ_{\text{lit}} \neg B$ if $A \succ_{\text{var}} B$,
  - $\neg A \succ_{\text{lit}} B$ if $A \succ_{\text{var}} B$ or $A = B$.
  - $\neg A \succ_{\text{lit}} \neg B$ if $A \succ_{\text{var}} B$, and

  Then $\succ_{\text{lit}}$ is total and well-founded.
- Define $\succ$ on clauses as the multiset extension of $\succ_{\text{lit}}$.
  Then $\succ$ is total and well-founded.
- Example: If $A \succ_{\text{var}} B$ then $\neg A \vee \neg B \vee B \succ \neg A \vee B \vee B \succ A \vee A \vee \neg B \vee \neg B$.

[a] Such an ordering always exists.

# Productive Clauses

Given an interpretation $I$, a clause $C$ and a propositional variable $A$, we say that the clause $C \vee A$ **produces** $A$ in $I$ iff $A \succ C$ and $I \not\models C \vee A$.

In words, a clause produces a variable $A$ in an interpretation if $A$ is the maximal literal in the clause and the clause is not yet true in the interpretation.

Examples: $(A \succ_{\text{var}} B)$

- $\neg A \vee \neg B \vee B$ does not produce anything.
- $A \vee B \vee B$ produces $A$ in every interpret. $I$ with $I(A) = I(B) = 0$.
- $A \vee A \vee B$ does not produce anything.

# Candidate Interpretation Construction

To show refutational completeness, we construct a so-called *candidate* interpretation $I(\Gamma)$ of a saturated set of clauses $\Gamma$. This construction succeeds whenever $\bot \notin \Gamma$.

- Let $\mathcal{A}$ be the (finite) set of propositional variables occuring in $\Gamma$.
- Define $I(\Gamma) := I_{\mathcal{A}}(\Gamma)$, and define $I_{\mathcal{A}}(\Gamma)$ inductively by:
  - $I_{\emptyset} := \emptyset$
  - If $I_X(\Gamma)$ is defined for $X \subseteq \mathcal{A}$ and $A$ is minimal in $\mathcal{A} \setminus X$ then

$$I_{X \cup \{A\}}(\Gamma) := \begin{cases} I_X \cup \{A\} & \text{if } \exists C \in \Gamma : C \text{ produces } A \text{ in } I_X \\ I_X & \text{otherwise} \end{cases}$$

Note that $I_{\mathcal{A}}$ is well-defined[a] because $\succ$ is total and well-founded.

[a] We identify an interpretation $I$ with the set $I^{-1}(1)$.

# Interpretation Existence Theorem

**Theorem:** Let $\Gamma$ be a saturated set of clauses. If $\bot \notin \Gamma$ then $I(\Gamma) \models \Gamma$.

**Proof (Bachmair, Ganzinger 1990):**

Towards a contradiction suppose that $\bot \notin \Gamma$ and $I(\Gamma) \not\models \Gamma$.

- As $\succ$ is a well-order ($=$ well-founded strict total order) on clauses, $I(\Gamma) \not\models \Gamma$ implies that there is a unique minimal clause $C \in \Gamma$ such that $I(\Gamma) \not\models C$.
- As $\bot \notin \Gamma$, $C \neq \bot$, so there is a unique literal $L \in C$ which is maximal, i. e., there is no $L' \in C$ with $L' \succ L$.
- As $I(\Gamma) \not\models C$, $C$ is not productive, i. e., the maximal literal $L$ either must be negative or occur at least twice.

# Interpretation Existence Theorem (cont'd)

**Proof:** (cont'd)

- 1st case: $L = \neg A$ for some variable $A$, i. e., $C = \neg A \vee C'$.
  - As $I(\Gamma) \not\models C$, $I(\Gamma) \models A$ and $I(\Gamma) \not\models C'$.
  - As $I(\Gamma) \models A$, there is some clause $D \in \Gamma$ which produces $A$, i. e., $D = D' \vee A$ and $A \succ D'$ and $I(\Gamma) \not\models D'$.
  - As $\Gamma$ saturated, resolvent $D' \vee C'$ of $D$ and $C$ is in $\Gamma$.
  - As $I(\Gamma) \not\models \{C', D'\}$, $I(\Gamma) \not\models D' \vee C'$.
  - As $C \succ C'$ and $C \succeq \neg A \succ A \succ D'$, $C \succ D' \vee C'$.
  - Contradition to minimality of $C$.

# Interpretation Existence Theorem (cont'd)

**Proof:** (cont'd)

- 2nd case: $L = A$ for some variable $A$, i. e., $C = C' \vee A \vee A$.
  - As $\Gamma$ saturated, factor $C' \vee A$ of $C$ is in $\Gamma$.
  - As $I(\Gamma) \not\models C$, $I(\Gamma) \not\models C' \vee A$.
  - Obviously, $C \succ C' \vee A$.
  - Contradition to minimality of $C$.

**Corollary:** If $\Gamma$ is unsatisfiable then $\perp \in \mathrm{Res}^*(\Gamma)$.