Sequent Calculus for Intuitionistic Logic

Equivalence of Sequent Calculus and Natural Deduction for intuitionistic logic

## Sequent Calculus for Inuitionistic Logic:

Sequent Calculus for intuitionistic logic can be obtained from sequent calculus for classical logic by restricting the size of the right hand side:

An intuitionistic sequent is an object of form

$$\Gamma \vdash \Delta, \text{ with } |\Delta| \leq 1.$$

So, the multiset $\Delta$ is either empty, or it contains one formula.

It is possible to further restrict intuitionistic sequents to form $\Gamma \vdash \Delta,$ with $|\Delta| = 1$. A sequent of form $\Gamma \vdash$ can be replaced by $\Gamma \vdash \bot$, so no generality is lost. Here, we keep the sequents with empty right hand side.

The rules for intuitionistic sequent calculus are (almost) the same as the rules for sequent calculus for classical logic.

Propositional Rules:

$$\text{(axiom)} \quad \frac{}{\Gamma, A \vdash A}$$

$$\text{(cut0)} \quad \frac{\Gamma, A \vdash \qquad \Gamma \vdash A}{\Gamma \vdash}$$

$$\text{(cut1)} \quad \frac{\Gamma, A \vdash B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

The cut1-rule is not a restriction of the cut rule for classical logic, because the premisses have different $\Delta$. However, one can reformulate cut for classical logic as

$$\text{(cut)} \quad \frac{\Gamma_1, A \vdash \Delta_1 \qquad \Gamma_2 \vdash A, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

Now, cut1 is a real subcase of the cut for classical logic.

Structural Rules:

$$\text{(weakening left)}\ \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \text{(weakening right)}\ \frac{\Gamma \vdash}{\Gamma \vdash A}$$

$$\text{(contraction left)}\ \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}$$

(There is no contraction on the right:

$$\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ )$$

Rules for the constants:

$$(\top\text{-left}\ )\ \frac{\Gamma \vdash \Delta}{\Gamma, \top \vdash \Delta} \qquad (\top\text{-right}\ )\ \frac{}{\Gamma \vdash \top}$$

$$(\bot\text{-left}\ )\ \frac{}{\Gamma, \bot \vdash \Delta} \qquad (\bot\text{-right}\ )\ \frac{\Gamma \vdash}{\Gamma \vdash \bot}$$

Rules for $\neg$:

$$(\neg\text{-left }) \ \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \qquad\qquad (\neg\text{-right }) \ \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}$$

Rules for $\wedge$ :

$$(\wedge\text{-left }) \ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad\qquad (\wedge\text{-right }) \ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

Rules for $\vee$ :

$$(\vee\text{-left})\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}$$

$$(\vee\text{-right1})\ \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad\qquad (\vee\text{-right2})\ \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

The rules $\vee$-right12 are not a restriction of the $\vee$-right rule for classical logic. The original form $\Gamma \vdash A, B \ \Rightarrow\ \Gamma \vdash A \vee B$ has too many conclusions on the right in the premisse. One can replace the $\vee$-right rule for classical logic by the following two rules:

$$\Gamma \vdash \Delta, A \ \Rightarrow\ \Gamma \vdash \Delta, A \vee B$$

and

$$\Gamma \vdash \Delta, B \ \Rightarrow\ \Gamma \vdash \Delta, A \vee B.$$

Rules for $\rightarrow$:

$$(\rightarrow\text{-left0}) \ \frac{\Gamma \vdash A \qquad \Gamma, B \vdash}{\Gamma, A \rightarrow B \vdash} \qquad\qquad (\rightarrow\text{-left1}) \ \frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \rightarrow B \vdash C}$$

$$(\rightarrow\text{-right}) \ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$$

(As with cut, the $\rightarrow$-left rule for classical logic has to be modified into

$$(\rightarrow\text{-left}) \ \frac{\Gamma_1 \vdash \Delta_1, A \qquad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \rightarrow B \vdash \Delta_1, \Delta_2}$$

so that $\rightarrow$-left12 are restrictions of it.)

Rules for $\leftrightarrow$::

$$(\leftrightarrow\text{-left}\ )\ \frac{\Gamma, A \rightarrow B,\ \ B \rightarrow A \vdash \Delta}{\Gamma, A \leftrightarrow B \vdash \Delta}$$

$$(\leftrightarrow\text{-right}\ )\ \frac{\Gamma \vdash A \rightarrow B \qquad \Gamma \vdash B \rightarrow A}{\Gamma \vdash A \leftrightarrow B}$$

Rules for the quantifiers:

$$(\forall\text{-left }) \ \frac{\Gamma, P[x := t] \vdash \Delta}{\Gamma, \forall x{:}\, X \ \ P(x) \vdash \Delta} \qquad (\forall\text{-right }) \ \frac{\Gamma \vdash P[x := y]}{\Gamma \vdash \forall x{:}\, X \ \ P(x)}$$

$$(\exists\text{-left }) \ \frac{\Gamma, P[x := y] \vdash \Delta}{\Gamma, \exists x{:}\, X \ \ P(x) \vdash \Delta} \qquad (\exists\text{-right }) \ \frac{\Gamma \vdash P[x := t]}{\Gamma \vdash \exists x{:}\, X \ \ P(x)}$$

The $t$ is an arbitrary term of type $X$. The $y$ is a variable of type $X$, which is not free in $\Gamma, \Delta$.

rules for equality:

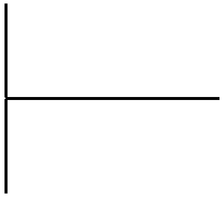
$$\text{(refl)} \ \frac{}{\Gamma \vdash t = t}$$

$$\text{(repl)} \ \frac{t_1 = t_2,\ \ \Gamma[t_2] \vdash \Delta[t_2]}{t_1 = t_2, \Gamma[t_1] \vdash \Delta[t_1]}$$

The last rule means: If $t_1 = t_2$ appears among the premisses, then arbitrary occurrences of $t_2$ in other formulas can be replaced by $t_1$.

## Some Anatomy of Natural Deduction Proofs

An object of form



is called a scope. The part above the line is the assumption part. It consists of formulas only. The part below the line is the derivation part. The derivation part consists of formulas and other scopes.

If an formula $F$ is assumed in the assumption part of some scope $S$, then all formulas in the derivation part of $S$ and in the derivation parts of scopes inside $S$ are derived on the assumption of $F$.

## Examples

$A_1$

$A_2$

$A_3$

$A_4$

$A_5$

$A_6$

$A_7$

$A_8$

$A_9$

$A_6$ is derived under the assumptions $A_1, A_2$. Formula $A_8$ is derived under the assumptions $A_1, A_2, A_7$.

## A More Practical Notation for ND-Proofs

Since ND-proofs are rather space consuming, we introduce a new notation for ND-proofs. We write a scope

$$\begin{array}{|l} A_1 \\ A_2 \\ \hline F_1 \\ F_2 \end{array}$$

as $[A_1, A_2 \mid F_1, F_2]$.

Theorem: Intuitionistic Sequent Calculus and Natural Deduction are Equally Strong. More precisely:

1. (a) If $\Gamma \vdash$ has a proof in intuitionistic sequent calculus, then one can construct a natural deduction proof of form $[\ \Gamma \mid \pi_1, \ldots, \pi_n, \perp\ ]$.

   (b) If $\Gamma \vdash A$ has a proof in intuitionistic sequent calculus, then one can construct a natural deduction proof of form $[\ \Gamma \mid \pi_1, \ldots, \pi_n, A\ ]$.

2. For every natural deduction proof in which a formula $A$ is derived under assumptions $\Gamma_1, \ldots, \Gamma_n$, the sequent $\Gamma_1, \ldots, \Gamma_n \vdash A$ has a proof in intuitionistic sequent calculus.

Lemma (Weakening):
If there is a natural deduction proof of form

$$[\ \Gamma \mid \pi_1, \ldots, \pi_n, B\ ]$$

then there exists a natural deduction proof of form

$$[\ \Gamma, A \mid \rho_1, \ldots, \rho_n, B\ ].$$

This rose has some thorns. The problem is caused by new names, required by $\exists$-elim and $\forall$-intro. If one allows the new name to hide other uses of the same name, then one has no problem. If one requires the new name to be really new, then one has a problem if it occurs in $A$. One has to show that the new name can be freely replaced by another name that does not occur in $A$. This is intuitively obvious, but technically involved. We omit the proof.

Now back to the theorem. 1a and 1b have to be proven together by induction on the structure of the sequent calculus proof. We go through the possible rules occurring in the sequent calculus proof. I do not discuss all rules, because the pattern is the same for all of them.

Consider the axiom $\Rightarrow \Gamma, A \vdash A$. One can construct the following natural deduction proof:

$$[\Gamma, A \mid A].$$

Consider the cut0-rule $\Gamma, A \vdash, \quad \Gamma \vdash A, \quad \Rightarrow \Gamma \vdash$
By induction there are natural deduction proofs

$$[\Gamma, A \mid \pi_1, \ldots, \pi_n, \perp], \text{ and}$$

$$[\Gamma \mid \rho_1, \ldots, \rho_m, A].$$

Then one can construct:

$$[\Gamma \mid \rho_1, \ldots, \rho_m, A, \pi_1, \ldots, \pi_n, \perp].$$

Consider the cut1-rule $\Gamma, A \vdash B, \quad \Gamma \vdash A, \quad \Rightarrow \Gamma \vdash B$.
By induction there are natural deduction proofs

$$[\Gamma, A \mid \pi_1, \ldots, \pi_n, A], \text{ and}$$

$$[\Gamma \mid \rho_1, \ldots, \rho_m, A].$$

Then one can construct:

$$[\Gamma \mid \rho_1, \ldots, \rho_m, A, \pi_1, \ldots, \pi_n, B].$$

Consider the $\wedge$-left rule $\Gamma, A, B \vdash C \quad \Rightarrow \Gamma, A \wedge B \vdash C$.
By induction, there is a natural deduction proof

$$[\Gamma, A, B \mid \pi_1, \ldots, \pi_n, C].$$

From this, one can construct:

$$[\Gamma, A \wedge B \mid A, B, \pi_1, \ldots, \pi_n, C].$$

Consider the $\wedge$-right rule $\Gamma \vdash A, \quad \Gamma \vdash B \quad \Rightarrow \Gamma \vdash A \wedge B$.
By induction, there are natural deduction proofs

$$[\Gamma \mid \pi_1, \ldots, \pi_n, A], \text{ and } [\Gamma \mid \rho_1, \ldots, \rho_m, B].$$

Then one can construct the following natural deduction proof

$$\Gamma \mid \pi_1, \ldots, \pi_n, A, \rho_1, \ldots, \rho_m, B, A \wedge B].$$

Consider the $\vee$-left rule $\Gamma, A \vdash C, \quad \Gamma, B \vdash C \Rightarrow \Gamma, A \vee B \vdash C$.
By induction, there are natural deduction proofs

$$[\Gamma, A \mid \pi_1, \ldots, \pi_n, C], \text{ and}$$

$$[\Gamma, B \mid \rho_1, \ldots, \rho_m, C].$$

From this one can construct the following proof:

$$[\Gamma, A \vee B \mid [A \mid \pi_1, \ldots, \pi_n, C],\ [B \mid \rho_1, \ldots, \rho_m, C],\ C].$$

Consider the $\forall$-left rule $\Gamma, P[x := t] \vdash A \quad \Rightarrow \; \Gamma, \forall x{:}X \; P(x) \vdash A$. By induction, one can assume that there is a natural deduction proof

$$[\ \Gamma, P[x := t] \mid \pi_1, \ldots, \pi_n, A],$$

From this, one can construct

$$[\ \Gamma, \forall x{:}X \; P(x) \mid P[x := t], \pi_1, \ldots, \pi_n, A].$$

Now consider the $\forall$-right rule $\Gamma \vdash P[x := y] \quad \Rightarrow \; \Gamma \vdash \forall x{:}X \; P(x)$. By induction, one can construct the proof $[\Gamma \mid \pi_1, \ldots, \pi_n, P[x := y]\ ]$. Because $y$ does not occur in $\Gamma$, one can construct the natural deduction proof

$$[\Gamma \mid \pi_1, \ldots, \pi_n, P[x := y], \forall x{:}X \; P(x)].$$

Consider the $\exists$-left rule $\Gamma, P[x := y] \vdash A \quad \Rightarrow \ \Gamma, \exists x{:}X \ P(x) \vdash A$. By induction, one can assume that there is a natural deduction proof

$$[\ \Gamma, P[x := y] \mid \pi_1, \ldots, \pi_n, A],$$

From this, one can construct

$$[\ \Gamma, \exists x{:}X \ P(x) \mid [P[x := y], \pi_1, \ldots, \pi_n, A\ ], A\ ]$$

Now consider the $\exists$-right rule $\Gamma \vdash P[x := t] \quad \Rightarrow \ \Gamma \vdash \exists x{:}X \ P(x)$. By induction, one can construct the proof $[\Gamma \mid \pi_1, \ldots, \pi_n, P[x := t]\ ]$. Because $y$ does not occur in $\Gamma$, one can construct the natural deduction proof

$$[\ \Gamma \mid \pi_1, \ldots, \pi_n, P[x := t], \exists x{:}X \ P(x)\ ].$$

We now prove the other side of the theorem by induction.

Assume that $A$ is obtained from $A \wedge B$ in a natural deduction proof, under assumptions $\Gamma$. By induction, one can obtain a sequent proof for $\Gamma \vdash A \wedge B$. Since the sequent $A \wedge B \vdash A$ is provable, we can use the cut1-rule to obtain $\Gamma \vdash A$.

Now assume that $A \wedge B$ is obtained from $A$ and $B$ under assumptions $\Gamma$ in a natural deduction proof. By induction, one may assume that there are proofs of $\Gamma \vdash A$ and $\Gamma \vdash B$. One these, one can apply $\wedge$-right to obtain $\Gamma \vdash A \wedge B$.

There is a general pattern: Introduction rules can remain introduction rules. Elimination rules need to be replaced by the cut1-rule.

Assume that $C$ is obtained from $A \vee B$ in a natural deduction proof, under assumptions $\Gamma$. So, we have a proof of form

$$[\ \cdots, [\ \ldots, A \vee B, [A \mid\ \cdots, C\ ],\ [A \mid\ \cdots, C\ ], C\ ] \cdots].$$

By induction, one can obtain sequent proofs for
$\Gamma \vdash A \vee B, \quad \Gamma, A \vdash C$, and $\Gamma, B \vdash C$. Applying $\vee$-left on the last two sequents, one obtains $\Gamma, A \vee B \vdash C$. Applying cut1, one obtains $\Gamma \vdash C$.

Now assume that $A \vee B$ is obtained from $A$ under assumptions $\Gamma$ in a natural deduction proof. By induction, one may assume that there is a proof of $\Gamma \vdash A$. One can apply $\vee$-right to obtain $\Gamma \vdash A \vee B$.

(The case where $A \vee B$ is derived from $B$ goes analogeously)

Assume that $B$ is obtained from $A \to B$ and $A$ in a natural deduction proof under assumptions $\Gamma$.

By induction, one can obtain sequent proofs for $\Gamma \vdash A \to B$ and $\Gamma \vdash A$. The sequent $A, A \to B \vdash B$ is provable. Applying cut1 two times, one obtains $\Gamma \vdash B$.

Now assume that $A \to B$ is obtained from the $\to$-introduction rule as follows:

$$[\ \cdots, [\ \ldots, [A \mid\ \cdots, B\ ],\ A \to B\ ] \cdots\ ].$$

Assume that the total context equals $\Gamma$. By induction, one may assume that there is a proof of $\Gamma, A \vdash B$. One can apply $\to$-right to obtain $\Gamma \vdash A \to B$.

Assume that $P(x)[x := t]$ is obtained from $\forall x{:}X\ P(x)$ under assumptions $\Gamma$.

By induction, one can obtain a sequent proof for $\Gamma \vdash \forall x{:}X\ P(x)$. The sequent $\forall x{:}X\ P(x) \vdash P(x)[x := t]$ is provable. Applying cut1, one obtains $\Gamma \vdash P(x)[x := t]$

Now assume that $\forall x{:}X\ P(x)$ is derived from $P(x)[x := t]$ in context $\Gamma$ in some natural deduction proof. By induction, the sequent $\Gamma \vdash P(x)[x := t]$ is provable. By the conditions of the $\forall$-introduction rule, $x$ does not occur in $\Gamma$. Hence, the $\forall$-right rule can be applied, and one obtains a proof of $\Gamma \vdash \forall x{:}X\ P(x)$.

Assume that $C$ is obtained from $\exists x{:}X\ P(x)$ in a natural deduction proof, under assumptions $\Gamma$. So, we have a proof of form

$$[\ \cdots, [\ \ldots, \exists x{:}X\ P(x), [P(x)[x := y] \mid \cdots, C\ ],\ C\ ] \cdots].$$

By induction, one can obtain sequent proofs for $\Gamma \vdash \exists x{:}X\ P(x)$ and $\Gamma, P(x)[x := y] \vdash C$. Applying $\vee$-left on the last two sequents, one obtains $\Gamma, A \vee B \vdash C$. Applying cut1, one obtains $\Gamma \vdash C$.

Now assume that $A \vee B$ is obtained from $A$ under assumptions $\Gamma$ in a natural deduction proof. By induction, one may assume that there is a proof of $\Gamma \vdash A$. One can apply $\vee$-right to obtain $\Gamma \vdash A \vee B$.

(The case where $A \vee B$ is derived from $B$ goes analogeously)

## Conclusions

This shows that intuitionistic sequent calculus and natural deduction have equal strength.