Natural Deduction for Intuitionistic Logic

There are several types of deduction systems. The two most important are:

- Natural Deduction: Natural Deduction tries to follow the natural style of reasoning. Most of the proof consists of forward reasoning. Occassionally assumptions for hypothetical reasoning. Natural Deduction fits better to intuitionistic logic than to classical logic.
- Sequent Calculus: In sequent calculus, conclusions and premisses are treated equally. This is more unnatural for human readers, but technically it is simpler. Automated proof search becomes easier. Sequent calculus fits naturally both to classical and to intuitionistic logic.

Intuitionistic Logic is obtained from classical logic by dropping the laws of excluded middle ($A \vee \neg A$) and the law of double negation ($\neg\neg A \to A$)

Intuitionistic Logic has the construction or witness property. That means: If one proves a statement of form $\forall x{:}X \ A(x) \vee B(x)$, then one can determine for every t, which of $A(t) \vee B(t)$ holds. Similarly, if one has proven a formula of form $\forall x{:}X \ y{:}Y \ p(x, y)$, one determine for every term t, the term u for which $p(t, u)$ is provable. This means that in intuitionistic logic, formulas correspond to datatypes, and proofs correspond to algorithms. This correspondence is called the Curry-Howard isomorphism. It is exploited in the Coq-system. We come to that later.

Natural Deduction proofs have form $\begin{array}{|l} \text{Assumptions} \\ \hline \text{Conclusions.} \end{array}$

The assumptions have to be formulas. The conclusions are also formulas. It is possible to introduce more assumptions in the conclusion-part of the proof. In this case, one obtains a nested proof, as for example in

$$\begin{array}{|l} \text{Assumptions} \\ \hline \text{Conclusions.} \\ \begin{array}{|l} \text{More assumptions} \\ \hline \text{More conclusions.} \end{array} \\ \text{More assumptions} \rightarrow \text{More conclusions.} \end{array}$$

The conclusions are derived from the assumptions using forward reasoning rules. For example if one has derived A and one has derived B, then one may conclude $A \wedge B$ and add this formula to the conclusions.

Natural Deduction for Intuitionistic Logic

Rules for the constants:

$\top$-introduction:

$$\begin{array}{|l} \ldots \\ \top \end{array}$$

$\bot$-elimination:

$$\begin{array}{|l} \bot \\ \ldots \\ A \end{array}$$

Rules for $\neg$:

$\neg$-introduction:

A

$\dots$

$\perp$

$\neg A$

$\neg$-elimination:

A

$\dots$

$\neg A$

$\dots$

$\perp$

Rules for $\wedge$:

$\wedge$-elimination:

$$\begin{array}{|l} A \wedge B \\ \dots \\ A \end{array} \qquad \begin{array}{|l} A \wedge B \\ \dots \\ B \end{array}$$

$\wedge$-introduction:

$$\begin{array}{|l} A \\ \dots \\ B \\ \dots \\ A \wedge B \end{array}$$

Rules for $\vee$:

$\vee$-elimination: (reasoning by cases)

$$
\begin{array}{|l}
A \vee B \\
\ldots \\
\quad\begin{array}{|l}
A \\ \hline
\ldots \\
C
\end{array} \\
\ldots \\
\quad\begin{array}{|l}
B \\ \hline
\ldots \\
C
\end{array} \\
C
\end{array}
$$

Rules for $\vee$ (continued):

$\vee$-introduction(left and right):

A	B
$\ldots$	$\ldots$
$A \vee B$	$A \vee B$

Rules for $\leftrightarrow$:

$\leftrightarrow$-elimination:

$$
\begin{array}{|l}
A \leftrightarrow B \\
\ldots \\
A \rightarrow B
\end{array}
\qquad\qquad
\begin{array}{|l}
A \leftrightarrow B \\
\ldots \\
B \rightarrow A
\end{array}
$$

$\leftrightarrow$-introduction:

$$
\begin{array}{|l}
A \rightarrow B \\
\ldots \\
B \rightarrow A \\
\ldots \\
A \leftrightarrow B
\end{array}
$$

Rules for $\longrightarrow$:

$\longrightarrow$-elimination:

$$\begin{array}{|l} A \\ \dots \\ A \rightarrow B \\ \dots \\ B \end{array}$$

$\longrightarrow$-introduction:

$$\begin{array}{|l} \begin{array}{|l} A \\ \hline \dots \\ B \end{array} \\ A \rightarrow B \end{array}$$

Rules for $\forall$:

$\forall$-elimination:

$\forall x{:}X\ P(x)$

$\ldots$

$P[x := t]$

$\forall$-introduction:

x is new object of type X

$\ldots$

$P(x)$

$\forall x{:}X\ P(x)$

Rules for $\exists$:

$\exists$-elimination:

$\exists x{:}\, X\ \ P(x)$

$\ldots$

x is new object of type X

$P(x)$

$\ldots$

C

C

(C must not contain x)

Rules for $\exists$ (continued): $\exists$-introduction:

$P[x := t]$

$\dots$

$\exists x{:}X\ \ P(x)$

Rules for $\approx$:

Equality reflexivity:

$$
\begin{array}{|l}
\ldots \\
t \approx t
\end{array}
$$

Equality replacement:

$$
\begin{array}{|l}
t_1 \approx t_2 \\
\ldots \\
P(t_1) \\
\ldots \\
P(t_2)
\end{array}
$$

Remarks:

Small variations in the rules are possible. For example $\vee$-elim, $\exists$-elim, $\neg$-rules.

All operations take place on toplevel. Logic is not Algebra! Equality is not Equivalence!
(This means that one cannot exchange $A = B$ and $A \leftrightarrow B$)

Natural deduction naturally belongs to intuitionistic logic. Sequent calculus corresponds to classical logic, but it can be easily adapted to intuitionistic logic.