The exercises 8.1 and 8.2 belong to our project of formalizing and verifying a propositional resolution calculus in PVS.

**Exercise 8.1** Clause ordering.

1. Extend the **multisets** library with a theory defining the multiset extension \( \succ_{\text{mul}} \) (see slide 2) of a partial order \( \succ \) which is given as a parameter. Formalize the properties of \( \succ_{\text{mul}} \): It is a partial order, which extends well-foundedness and totality of \( \succ \) to multisets.
   
   **Hint:** Proving these properties is tedious; it suffices to formalize them as axioms.

2. Prove the following lemma:

   For all multisets \( S, T \) and elements \( x \),
   
   \[
   \text{if } x \in T \text{ and } \{x\} \succ_{\text{mul}} S \text{ then } T \succ_{\text{mul}} S \cup (T \setminus \{x\}).
   \]

   Point out where this lemma will be needed in the completeness proof.

3. Extend the **clauses** library with a theory defining the clause ordering \( \succ \) (see slide 3).
   
   **Hint:** Assert the existence of a well-order on the propositional variables by an axiom.

**Exercise 8.2** Candidate interpretation.

1. Formalize the ternary relation *a clause \( C \) produces a propositional variable \( A \) in an interpretation \( I \) (see slide 4) in PVS.

2. Formalizing the inductive definition of \( I_X(\Gamma) \) from slide 5 in PVS. and define the *candidate interpretation* \( I(\Gamma) \).

3. Prove the following monotonicity lemma for all sets \( X \) and \( Y \) such that \( I_X(\Gamma) \) and \( I_Y(\Gamma) \) are defined:

   \[
   \text{If } X \subseteq Y \text{ and } I_X(\Gamma) \models C \text{ then } I_Y(\Gamma) \models C.
   \]

   Point out where this lemma will be needed in the completeness proof.

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1The former statement \( \forall R, S, T : R \succ_{\text{mul}} S \land R \succ_{\text{mul}} T \Rightarrow R \succ_{\text{mul}} S \cup T \) is wrong.
Exercise 8.3 Proof terms and proof normalization via the Curry-Howard correspondence.

1. Construct the proof terms for the two proofs of $\varphi \vdash \varphi \rightarrow \psi \rightarrow \psi$ from slide 22.

2. $\beta$-reduce the longer proof term into the shorter one. Is the shorter proof term in $\beta$-normal form? Why?

Exercise 8.4 Natural deduction proofs via the Curry-Howard correspondence. Convert the type derivation trees of the combinator terms I, K, K* and S (see exercise 7.4) into natural deduction proofs of the combinators.

Exercise 8.5 Proof terms for minimal logic with conjunction and disjunction. Construct proof terms (i.e., inhabitants) for the following formulas:

1. $A \land B \rightarrow B \land A$.
2. $A \lor B \rightarrow B \lor A$.
3. $A \land (B \land C) \rightarrow (A \land B) \land C$.
4. $(A \land B) \land C \rightarrow A \land (B \land C)$.
5. $A \lor (B \lor C) \rightarrow (A \lor B) \lor C$.
6. $A \lor (B \land C) \rightarrow (A \lor B) \land (A \lor C)$.
7. $(A \land B) \lor (A \land C) \rightarrow A \land (B \lor C)$.

Challenge: Construct an inhabitant of 4 using (repeated) proofs of 3 and 1. Normalize.