Solution to exercises 4 and 5

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I illustrate the reductions from PCP to unsatisfiability and to existence of a finite model using an example instance of PCP. It should be not difficult to derive the general construction from that.
Consider the following PCP-instance

$$V = (a, bc, ba), \quad W = (ab, bc, a).$$

It has a single solution $(1, 3)$, which results in the word (aba).
Reduction to Unsatisfiability

If you know Prolog, or resolution, the construction is very easy to understand.

We use the following symbols:

- A constant λ, denoting the empty word.
- For each letter x in the alphabet, a unary function symbol x.
- A constant 0, and a unary function s.
- A ternary predicate symbol p.

Then we translate the problem into:

$p(\lambda, \lambda, 0)$,

$\forall xyn\ p(x, y) \rightarrow p(a(x), b(a(y)), s(n))$,
$\forall xyn\ p(x, y) \rightarrow p(c(b(x)), c(b(y)), s(n))$,
$\forall xyn\ p(x, y) \rightarrow p(a(b(x)), a(y), s(n))$,

$\forall xn\ \neg p(x, x, s(n))$.

First prove $p(\lambda, \lambda, 0)$, then $p(a(\lambda), b(a(\lambda)), s(0))$, then $p(a(b(a(\lambda))), a(b(a(\lambda))), s(s(0))$, and after that $\perp$.

Reduction to Existence of Finite Model
We use the following symbols:

- A unary predicate λ, denoting the empty word.
- For each letter x in the alphabet, a binary function symbol x.

- A binary predicate symbol p.
- We first set a starting point:

$$\exists x\ p(x,x) \wedge \lambda(x).$$

- Then comes the main translation:

  Let $\Phi(x,y) := p(x,y) \wedge (x \neq y \vee \lambda(x))$.

  Let $C_1(x,y) := \exists x_1 y_1 y_2\ a(x,x_1) \wedge a(y,y_1) \wedge b(y_1,y_2) \wedge p(y_1,y_2)$

  Let $C_2(x,y) := \exists x_1 x_2 y_1 y_2\ b(x,x_1) \wedge c(x_1,x_2) \wedge b(y,y_1) \wedge c(y_1,y_2) \wedge p(y_1,y_2)$.

  Let $C_3(x,y) := \exists x_1 x_2\ y_1\ b(x,x_1) \wedge a(x_1,x_2) \wedge a(y,y_1) \wedge p(x_2,y_1)$.

  The main translation is:

$$\forall xy\ \Phi(x,y) \rightarrow C_1(x,y) \vee C_2(x,y) \vee C_3(x,y)$$

  The idea of the construction is that, when $(c,d) \in [p]$, and $c \neq p$, we are forced to select one of the C_1, C_2, C_3 which will create two new points $(c', d') \in [p]$, s.t. c' is reachable through a chain from c, and d' is reachable trough a chain from d. The chains correspond to a pair (v_i, w_i) from the orginal PCP problem. The only way to stop extending forever, is to reach a point $(c'', d'') \in [p]$ with $c'' = d''$.

- And we ensure that every point that is reachable from a point x on which $\lambda(x)$ holds, is reachable in only one way:

  (Otherwise, we could for example start in a point (x,y) with $x = y$, make the chain $b(x,x_1), a(x_1,x_2)$ end in the same point as the chain $a(x,y_1)$, have $p(x_2,y_1)$, and stop by cheating)

  There is no way back to λ :

  $\forall x_1 x_2\ a(x_1,x_2) \wedge \lambda(x_2) \rightarrow \perp$,
  $\forall x_1 x_2\ b(x_1,x_2) \wedge \lambda(x_2) \rightarrow \perp$,
  $\forall x_1 x_2\ c(x_1,x_2) \wedge \lambda(x_2) \rightarrow \perp$.

  Through different predicates a, b, c one cannot reach the same point:
  $\forall xyz\ a(x,z) \wedge b(y,z) \rightarrow \perp$,
  $\forall xyz\ a(x,z) \wedge c(y,z) \rightarrow \perp$,
  $\forall xyz\ b(x,z) \wedge c(y,z) \rightarrow \perp$.

  Through each of a, b, c one cannot arrive at the same point from different start positions:
  $\forall x_1 x_2\ y_1 y_2\ \ a(x_1,x_2) \wedge a(y_1,y_2) \wedge x_2 = y_2 \rightarrow x_1 = y_1$,
  $\forall x_1 x_2\ y_1 y_2\ \ b(x_1,x_2) \wedge b(y_1,y_2) \wedge x_2 = y_2 \rightarrow x_1 = y_1$,
  $\forall x_1 x_2\ y_1 y_2\ \ c(x_1,x_2) \wedge c(y_1,y_2) \wedge x_2 = y_2 \rightarrow x_1 = y_1$.

In this case, there is a finite model, which can be described as follows:

$$D = \{d_0, d_1, d_2, d_3\},$$

$$
\begin{aligned}
[\lambda] &= \{d_0\}, \\
[A] &= \{(d_0, d_1), (d_2, d_3)\}, \\
[B] &= \{(d_1, d_2)\}, \\
[C] &= \{\}, \\
[p] &= \{(d_0, d_0), (d_1, d_2), (d_3, d_3)\}.
\end{aligned}
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