Mechanics of Rigid Objects

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1 Mechanics of Rigid Objects

Rigid objects are objects that do not significantly change their form, when forces are exerted on them.

2 Collections of Point Masses

The behaviour of rigid objects can be derived from the elementary laws of Newton mechanics.

Definition 2.1 *A* point mass M *is a triple* $(m, \overline{x}, \overline{v})$, *in which*

- $m \neq 0$ *is the mass.*
- $\overline{x} \in (\mathcal{R} \rightarrow \mathcal{R}^3)$ *is the position function. At each time* t, *the value* $\overline{x}(t)$ *defines the position of the point mass.*
- $\overline{v} \in (\mathcal{R} \rightarrow \mathcal{R}^3)$ *is the speed function. At each time* t, *the value* $\overline{v}(t)$ *defines the speed of the point mass.*

Movement of a point mass can be computed by Newton's law,

$$\forall t, \quad \overline{F}(t) = m.\ \overline{a}(t). \tag{1}$$

Since

$$\text{for all } t, \quad \overline{x}(t) = \overline{x}(t_0) + \int_{t_0}^{t} \overline{v}(u)\ du, \text{ and } \overline{v}(t) = \overline{v}(t_0) + \int_{t_0}^{t} \overline{v}(u)\ du, \tag{2}$$

the speed $\overline{v}(t)$ and $\overline{x}(t)$ can be integrated (for example, using our favourite method RK4) on the condition that $\overline{F}(t)$ is known.

We will usually omit the time parameter.

Another, extremely important property is the fact that forces can be added and decomposed, independent of their type. For example, if a point mass M_1 feels the gravity from two other point masses M_2 and M_3, the gravity forces originating from M_2 and M_3 can be computed independently, and added. Similarly, the forces on an electrically charged object with mass can be computed independently. One can first compute the electromagnetic force, ignoring the mass, and after that compute the gravity, ignoring the electric charge. The forces can be added to obtain the total forces. So we have: In every collection of point masses that does not exchange force with the outside world, all force can be decomposed into simple, direct forces between pairs of point masses. These simple forces are always reciprocal: The point masses experience the force in opposite direction, and directed along the unique straight line that goes through both of them. (This implies that there are only two possibilities: Either the point masses attract each other, or they repel each other.)

Definition 2.2 *An* object $\mathcal{O}$ *is a (finite) collection of point masses* $M_1, \ldots, M_n$. *The* total mass $m_{\mathcal{O}}$ *of* $\mathcal{O}$ *is defined as*

$$m_{\overline{\mathcal{O}}} = m_1 + m_2 + \cdots + m_n.$$

An external force working on $\mathcal{O}$ *is a pair of form* $(i, \overline{F})$, *where* $1 \leq i \leq n$, *and* $\overline{F} \in \mathcal{R}^3$.
An internal force of $\mathcal{O}$ *is a force between two point masses of* $\mathcal{O}$. *It can be represented by a triple of form* $(i, j, \overline{F})$, *where* $1 \leq i, j \leq n$, *and* $\overline{F} \in \mathcal{R}^3$.

$\overline{F}$ *must work along the line through* $\overline{x}_i$ *and* $\overline{x}_j$. *This means that there exists a* $\lambda \in \mathcal{R}$, *s.t.*

$$\overline{x}_i + \lambda \overline{F} = \overline{x}_j.$$

Internal forces are always between two mass points in the object. Both masses feel the same force, but in opposite direction. We will assume that the internal force $(i, j, \overline{F})$ specifies the force on M_i. The force on M_j will be $-\overline{F}$.

The gravity force, that we have seen in exercise list 2 is an internal force of the solar system.

Note that the distinction between internal and external forces is kind of arbitrary. An external force can always be made internal by finding its cause, and adding the cause to the object. For example, if one considers the system consisting of the moon and the earth alone separately, then the gravity from the sun is an external force. If one includes the sun into the system, it becomes an internal force.

Representing forces by pairs of form $(i, \overline{F})$ or $(i, j, \overline{F})$ turned out very inconvenient. For this reason, we introduce the following notions:

Definition 2.3 *Let* $\mathcal{O} = (M_1, \ldots, M_n)$ *be an object. Let* $(i, \overline{F})$ *be an external force working on* $\mathcal{O}$. *The* external force vector $\overline{E}(i, \overline{F})$ *of* $(i, \overline{F})$ *is defined by*

$$\begin{cases} \overline{E}_{i_1} = \overline{0} & \text{if } i_1 \neq i, \\ \overline{E}_i = \overline{F}. & \end{cases}$$

The external force vector $\overline{E}$ *of* $\mathcal{O}$ *is the sum of all external force vectors* $\overline{E}(i, \overline{F})$, *for all external forces working on* $\mathcal{O}$.

Let $(i, j, \overline{F})$ *be an internal force of* $\mathcal{O}$*. The* internal force matrix $\overline{I}(i, j, \overline{F})$ *is defined by the following properties:*

$$\begin{cases} \overline{I}_{i_1,j_1} = \overline{0} & \text{if } i_1 \neq i \text{ or } j_1 \neq j, \text{ and } i_1 \neq j \text{ or } i_1 \neq i, \\ \overline{I}_{i,j} = \overline{F} \\ \overline{I}_{j,i} = -\overline{F}. \end{cases}$$

The internal force matrix $\overline{I}$ *of* $\overline{O}$ *is the sum of all internal force matrices* $\overline{I}(i, j, \overline{F})$*, for all internal forces of* $\mathcal{O}$*.*

A *Rigid Object* is an object that is able to organize its internal forces in such a way that its point masses do not change their position, relative to each other. We will first derive a couple of general properties of objects, which are independent of rigidity.

3 Angular Speed, Angular Acceleration and Rigidity

We first define angular acceleration and angular speed. After that, define what a rigid object is: It is an object, for which the speed of its points is determined by the speed of a central point, and the amound of rotation around this point

Definition 3.1 Angular velocity *(also called* angular speed*)* $(\overline{c}, \overline{\omega})$ *is defined by a rotation center* $\overline{c}$*, and a rotation speed* $\overline{\omega}$*.*

The angular velocity a linear velocity $\overline{v}$ *to each point* $\overline{x}$ *as follows:*

$$\overline{v} = \overline{\omega} \times (\overline{x} - \overline{c}).$$

Angular acceleration $(\overline{c}, \overline{\alpha})$ *is defined similar to angular velocity. It consists of a rotation center* $\overline{c}$*, and an angular acceleration vector* $\overline{\alpha}$*. The angular acceleration* $(\overline{c}, \overline{\alpha})$ *assigns to each point a linear acceleration* $\overline{a}$ *as follows:*

$$\overline{a} = \overline{\alpha} \times (\overline{x} - \overline{c}).$$

It should be noted that in most cases $\overline{v}, \overline{a}, \overline{\omega}, \overline{\alpha}$ and $\overline{c}$ are functions of t.
We can now define what is a rigid object:

Definition 3.2 *Let* $\mathcal{O} = (M_1, \ldots, M_n)$ *be an object. We call* $\mathcal{O}$ rigid *if there exist a time dependent position* $\overline{x}$*, a time dependent linear speed* $\overline{v}$*, and a time dependent angular speed* $\overline{\omega}$*, s.t always, for every mass point* $(m_i, \overline{x}_i, \overline{v}_i)$ *of* $\mathcal{O}$*,*

$$\overline{v}_i = \overline{v} + \overline{\omega} \times (\overline{x}_i - \overline{x}).$$

In words: An object is rigid if each moment in time, the speed of each of its points can be obtained by adding the linear speed of a reference point of the object and the speed caused by some angular speed around this reference point.

4 Loosely Connected Objects

Before we will study rigid objects in more detail, we will first have a closer look at objects that are not rigid. Let us call these objects loosely connected objects.

4.1 Linear Properties

Definition 4.1 *Let $\mathcal{O} = M_1, \ldots, M_n$ be an object.*
The center of mass $\overline{C}_{\mathcal{O}}$ *of $\mathcal{O}$ is defined as*

$$\frac{\sum_{i=1}^{n} m_i \overline{x}_i}{m_{\mathcal{O}}}.$$

The average speed $\overline{V}_{\mathcal{O}}$ *of $\mathcal{O}$ is defined as*

$$\frac{\sum_{i=1}^{n} m_i \overline{v}_i}{m_{\mathcal{O}}}.$$

Definition 4.2 *Let $\mathcal{O}$ be an object. Let $\overline{E}$ be an external force vector. The* total force $\overline{F}(\overline{E})$ *of $\overline{E}$ on $\mathcal{O}$ is defined as*

$$\overline{F}(\overline{E}) = \sum_{i=1}^{n} \overline{E}_i.$$

The following property states that if one has a loosely connected object, with external forces $\overline{E}$ working on it, then its center of mass of $\mathcal{O}$ accelerates

Theorem 4.3 *Let $\mathcal{O} = (M_1, \ldots, M_n)$ be an object. Let $\overline{E}$ be the external force vector. Then*

$$\overline{F}(\overline{E}) = m_{\overline{O}}(\overline{V}_{\mathcal{O}})'.$$

Proof
The main insight is to see that internal forces cancel each other. This a consequence of the fact that internal forces are always symmetric: $\overline{I}_{i,j} = -\overline{I}_{j,i}$. The symmetry implies that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} I_{i,j} = \overline{0}. \tag{3}$$

For each point mass M_i, we have

$$\overline{F}_i = m_i \overline{a}_i = m_i \overline{v}_i'.$$

Summation results in

$$\sum_{i=1}^{n} \overline{F}_i = \sum_{i=1}^{n} m_i \overline{v}_i'.$$

Using (3), we obtain

$$\overline{F}(\overline{E}) = \sum_{i=1}^{n} m_i \overline{v}_i'.$$

The right hand side can be multiplied and divided by $m_{\mathcal{O}}$. The result is

$$\overline{F}(\overline{E}) = m_{\mathcal{O}} \frac{\sum_{i=1}^{n} m_i \overline{v}_i'}{m_{\mathcal{O}}} = m_{\mathcal{O}} (\overline{V}_{\mathcal{O}})'.$$

Property 4.3 can be used to integrate the mass center of an object $\mathcal{O}$ without knowing anything about the internal forces of $\mathcal{O}$. In a rigid object, the mass center stays fixed at one place in the object.

The center of mass is often called center of gravity. In Section 4.3, I will explain why this makes sense.

4.2 Angular Properties: Torque and Angular Momentum

We will now define the angular counterparts of average speed and total force.

Definition 4.4 *Let $\overline{c}$ be a position. (Called* the reference position*) Let $M = (m, \overline{x}, \overline{v})$ be a point mass. The* angular momentum $L_M(\overline{c})$ *of M around $\overline{c}$ is defined as*

$$L_M(\overline{c}) = m(\overline{x} - \overline{c}) \times \overline{v}.$$

Let $\mathcal{O} = (M_1, \ldots, M_n)$ be an object. The angular momentum $\overline{L}_{\mathcal{O}}(\overline{c})$ *of $\mathcal{O}$ around $\overline{c}$ is defined as*

$$\overline{L}_{\mathcal{O}}(\overline{c}) = \sum_{i=1}^{n} m_i \, (\overline{x}_i - \overline{c}) \times \overline{v}_i.$$

Definition 4.5 *Let $\overline{F}$ be a force working on a point mass $(m, \overline{x}, \overline{v})$. Let $\overline{c}$ be a reference point. The* torque $\tau_M(\overline{F})$ *of $\overline{F}$ on M* around $\overline{c}$, *is defined as:*

$$\overline{\tau}_M(\overline{F}) = (\overline{x}_i - \overline{c}) \times \overline{F}.$$

Let $\overline{F}$ be a vector on forces on $\mathcal{O}$. Let $\overline{c}$ be a reference position. The torque of $\overline{F}$ *around $\overline{c}$ on $\mathcal{O}$ is defined as:*

$$\overline{\tau}_{\overline{c}}(\overline{F}) = \sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{F}_i.$$

The simple torque $\overline{\tau}$ is a function of $(i, \overline{F})$ and $\mathcal{O}$. In addition, it can also be a function of the time t. Physicists tend to omit dependencies and we follow this habit most of the time.

We will now prove an extremely important property, namely that the torques caused by internal forces in an object always cancel each other. We have seen already in the proof of Theorem 4.3 that internal forces cancel each other as linear forces. We will now see that they also cancel each other as angular forces.

Theorem 4.6 *Let $\mathcal{O}$ be an object. Let $\overline{I}$ be an internal force matrix of $\mathcal{O}$. Let $\overline{F}$ be the force vector, that attaches to each point mass of $\mathcal{O}$ the total result of the internal forces in $\mathcal{O}$, so each*

$$\overline{F}_i = \sum_{j=1}^{n} \overline{I}_{i,j}.$$

Let $\overline{c}$ be an arbitrary reference point. Then

$$\tau_{\overline{c}}(\overline{F}) = \overline{0}.$$

Proof
The torque $\tau_{\overline{c}}(\overline{F})$ equals the sum of its components, so that

$$\tau_{\overline{c}}(\overline{F}) = \sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{F}_i =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{I}_{i,j}. \tag{4}$$

Now comes the essential point: We already know that $\overline{I}_{i,j} = -\overline{I}_{i,j}$, but we will show that also the torques caused by $\overline{I}_{i,j}$ and $\overline{I}_{j,i}$ cancel each other. This is a consequence of the fact that the internal force between i and j is aligned along the vector $\overline{x}_j - \overline{x}_i$. (Each of the point masses feels a force that is directed exactly towards or away from the other point mass.) Because of the alignment, we can write $\overline{I}_{i,j}$ in the form $\lambda(\overline{x}_j - \overline{j})$, for some $\lambda \in \mathcal{R}$.
We also know that $I_{i,i} = \overline{0}$, because point masses have no internal force with themself, so that the summation (4) can be reorganized as

$$\sum_{i=1}^{n} \sum_{j=1}^{i-1} (\overline{x}_i - \overline{c}) \times \overline{I}_{i,j} + (\overline{x}_i - \overline{c}) \times \overline{I}_{j,i}.$$

We will show that for every $1 \leq i, j \leq n$,

$$(\overline{x}_i - \overline{c}) \times \overline{I}_{i,j} + (\overline{x}_i - \overline{c}) \times \overline{I}_{j,i} \tag{5}$$

is always equal to $\overline{0}$. Since $\overline{I}_{i,j} = \lambda(\overline{x}_j - \overline{x}_i)$ and $\overline{I}_{j,i} = \lambda(\overline{x}_i - \overline{x}_j)$, we can expand (5) into

$$\lambda(\overline{x}_i \times \overline{x}_j - \overline{x}_i \times \overline{x}_i - \overline{c} \times \overline{x}_j + \overline{c} \times \overline{x}_i + \quad \overline{x}_i \times \overline{x}_i - \overline{x}_i \times \overline{x}_j - \overline{c} \times \overline{x}_i + \overline{c} \times \overline{x}_j).$$

Because $\overline{x}_i \times \overline{x}_i = \overline{0}$, we see that the result is $\overline{0}$. That completes the proof.

Theorem 4.7 *Let $\overline{c}$ be a position that does not change through time. Let $\mathcal{O} = (M_1, \ldots, M_n)$ be a loosely connected object. Let $\overline{E}$ be the external force vector on $\mathcal{O}$, at some moment t. Let $\overline{I}$ be an internal force matrix for $\mathcal{O}$. Then*

$$\overline{\tau}_{\overline{c}}(\overline{E}) = [\ \overline{L}_{\mathcal{O}}(\overline{c})\ ]',$$

independent of the internal force matrix $\overline{I}$.

Proof
Newton's law holds for each point mass in $\mathcal{O}$:

$$\overline{F}_i = m_i \overline{a}_i = m_i \overline{v}'_i,$$

where $\overline{F}_i$ is the total force on M_i. Left-multiplying with $\overline{x}_i - \overline{c}$ results in

$$(\overline{x}_i - \overline{c}) \times \overline{F}_i = m_i (\overline{x}_i - \overline{c}) \times \overline{v}'_i.$$

Summation results in

$$\sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{F}_i = \sum_{i=1}^{n} m_i (\overline{x}_i - \overline{c}) \times \overline{v}'_i. \tag{6}$$

We will show that the left hand side is equal to $\overline{\tau}_{\overline{c}}(\overline{E})$, while the right hand side is equal to $[\overline{L}_{\mathcal{O}}(\overline{c})]'$. For the left hand side, each $\overline{F}_i$ consists of the external force on M_i, combined with the sum of the internal forces, so that $\overline{F}_i = \overline{E}_i + \sum_{j_1}^{n} \overline{I}_{i,j}$. It follows, using Definition 4.5 and Theorem 4.6, that

$$\sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{F}_i = \sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{E}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{I}_{i,j} = \tau_{\overline{c}}(\overline{E}).$$

In order to show that the right hand side of (6) equals $[\ \overline{L}_{\mathcal{O}}(\overline{c})\ ]'$, we compute

$$[\ \overline{L}_{\mathcal{O}}(\overline{c})\ ]' = [\ \sum_{i=1}^{n} m_i (\overline{x}_i - \overline{c}) \times \overline{v}_i\ ]' = \sum_{i=1}^{n} m_i (\overline{v}_i \times \overline{v}_i) + m_i (\overline{x} - \overline{c}) \times \overline{v}'_i.$$

Since $\overline{v}_i \times \overline{v}_i = \overline{0}$, this completes the proof that the right hand side of (6) equals $[\ \overline{L}_{\mathcal{O}}(\overline{c})\ ]'$.

4.3 Torque around Center of Gravity

The center of mass is often called the center of gravity. The reason is that fact that gravitation causes no torque around the center of gravity.

Theorem 4.8 *Let $\mathcal{O} = (M_1, \ldots, M_n)$ be a (loosely connected) object. Suppose that the internal force vector $\mathcal{E}$ is completely caused by gravity. Then $\overline{E}$ has form $(m_1 \overline{g}, \ldots, m_n \overline{g})$.*
Let $\overline{c}$ be the center of mass of $\mathcal{O}$. Then $\overline{\tau}_{\overline{c}}(\overline{E}) = \overline{0}$.

Proof
Using the definition of $\overline{\tau}_{\overline{c}}(\overline{E})$, we have

$$\overline{\tau}_{\overline{c}}(\overline{E}) = \sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times m_i \overline{g}$$

Using the distributive property, this is equal to

$$\sum_{i=1}^{n} (m_i \overline{x}_i \times \overline{g}) - \sum_{i=1}^{n} (m_i \overline{c} \times \overline{g}).$$

On the right hand side, we have

$$\sum_{i=1}^{n} (m_i \overline{c} \times \overline{g}) = (\sum_{i=1}^{n} m_i)(\overline{c} \times \overline{g}) = (\sum_{i=1}^{n} m_i \overline{x}_i) \times \overline{g},$$

so that the difference is indeed $\overline{0}$. In the last replacement, the definition of mass center was used.

Note that theorem 4.8 can be generalized to every point on the line $\overline{C}_{\mathcal{O}} + \lambda \overline{g}$. Let 'above' and 'below' be defined by the vector $\overline{g}$. Then gravity does not cause any torque in object $\mathcal{O}$ on every point that is exactly above or below the center of mass.

5 Movement Laws for Rigid Objects

Since rigid objects are a subset of non-rigid objects, all results Section 4.1 and Section 4.2 can be applied to rigid objects. In addition with Definition 3.2, one can specify how a rigid object reacts to an external force vector $\overline{E}$, independent of the internal force matrix $\overline{I}$.

Lemma 5.1 *Let $\overline{u}$ and $\overline{v}$ be two functions of type $\mathcal{R} \rightarrow \mathcal{R}^3$.*

$$(\overline{u} \times \overline{v})' = \overline{u} \times \overline{v}' + \overline{u}' \times \overline{v}.$$

Proof
It is the usual multiplication rule, and its proof is also usual.
We derive the motion laws for rigid objects. In this section, we give a purely formal definition, which may be hard to follow. I give an easier derivation in class.
By Definition 3.2, we have for each point mass M_i,

$$\overline{v}_i = \overline{v} + \overline{\omega} \times (\overline{x}_i - \overline{x}). \tag{7}$$

Differentation (using Lemma 5.1) results in

$$\overline{a}_i = \overline{a} + \overline{\alpha} \times (\overline{x}_i - \overline{x}) + \overline{\omega} \times (\overline{v}_i - \overline{v}).$$

The meaning of this last equation is that the linear acceleration of point mass M_i consists of three components: **(1)** The common linear acceleration of the rigid object $\mathcal{O}$, **(2)** the acceleration resulting from change of angular velocity, and **(3)** the acceleration caused by the rotation of the object, (centripetal acceleration). The term $\overline{v}_i - \overline{v}$ denotes the difference in speed between M_i and the reference center of the object. By using (7) another time, we have $(\overline{v}_i - \overline{v}) = \omega \times (\overline{x}_i - \overline{x})$, so that we obtain

$$\overline{a}_i = \overline{a} + \overline{\alpha} \times (\overline{x}_i - \overline{x}) + \overline{\omega} \times (\overline{\omega} \times (\overline{x}_i - \overline{x})).$$

Multiplying with the mass m_i and remembering that $\overline{F}_i = \overline{m}_i \overline{a}_i$ results in

$$\overline{F}_i = m_i \overline{a} + m_i \overline{\alpha} \times (\overline{x}_i - \overline{x}) + m_i \overline{\omega} \times (\overline{\omega} \times (\overline{x}_i - \overline{x})). \tag{8}$$

- In order to compute the linear behaviour of the rigid object, we add Equation 8 for the different i. The result (using 3 in the process) is

$$\overline{F}(\overline{E}) = \sum_{i=1}^{n} m_i \overline{a} + \sum_{i=1}^{n} m_i \overline{\alpha} \times (\overline{x}_i - \overline{x}) + \sum_{i=1}^{n} m_i \overline{\omega} \times (\overline{\omega} \times (\overline{x}_i - \overline{x})) =$$

  One can use the fact that

$$\sum_{i=1}^{n} m_i (\overline{x}_i - \overline{x}) = m_{\mathcal{O}} (\overline{C}_{\mathcal{O}} - \overline{x}).$$

  If one takes $\overline{x} = \overline{C}_{\mathcal{O}}$, this will be $\overline{0}$.

- In order to establish the relation between torque and angular acceleration, we left-multiply Equation 8 with the distance $\overline{x}_i - \overline{x}$ to the reference point $\overline{x}$. The result is $(\overline{x}_i - \overline{x}) \times \overline{F}_i =$

$$m_i (\overline{x}_i - \overline{x}) \times \overline{a} + m_i (\overline{x}_i - \overline{x}) \times (\overline{\alpha} \times (\overline{x}_i - \overline{x})) + m_i (\overline{x}_i - \overline{x}) \times (\overline{\omega} \times (\overline{\omega} \times (\overline{x}_i - \overline{x}))).$$

  The equations (for different i) can now be added, and one can use the fact that the sum of all torques inside the object equals the sum of the external torques working on the object, because internal torques cancel each other (Theorem 4.6):

$$\overline{\tau}_{\overline{x}}(\overline{E}) = \sum_{i=1}^{n} m_i (\overline{x}_i - \overline{x}) \times \overline{F}_i.$$

  The result is:

$$\overline{\tau}_{\overline{x}}(\overline{E}) = \sum_{i=1}^{n} m_i (\overline{x}_i - \overline{x}) \times \overline{a} + \sum_{i=1}^{n} m_i (\overline{x}_i - \overline{x}) \times (\overline{\alpha} \times (\overline{x}_i - \overline{x})) +$$

$$\sum_{i=1}^{n} m_i (\overline{x}_i - \overline{x}) \times (\overline{\omega} \times (\overline{\omega} \times (\overline{x}_i - \overline{x}))).$$

  The matrix M_i, defined by $M_i \alpha = m_i (\overline{x} - \overline{x}) \times (\overline{\alpha} \times (\overline{x} - \overline{x}))$ is called *the inertia matrix* of point mass M_i relative to $\overline{x}$. The matrix

$$\sum_{i=1}^{n} M_i$$

is called the *inertia matrix* of the object $\mathcal{O}$ relative to $\overline{x}$.

It turns out that for all vectors $\overline{v}$ and $\overline{w}$,

$$\overline{v} \times (\overline{w} \times (\overline{w} \times \overline{v})) = -\overline{w} \times (\overline{v} \times (\overline{v} \times \overline{w})).$$

(I know no better proof than writing it out, but maybe there exists a better explanation.) This implies that the main formula can be simplified into

$$\overline{\tau}_{\overline{x}}(\overline{E}) = m_{\mathcal{O}}(\overline{C}_{\mathcal{O}} - \overline{x}) \times \overline{a} + M(\overline{\alpha}) + \overline{\omega} \times M(\overline{\omega}).$$

I am not sure yet whether it is a good idea to take $\overline{x} = \overline{C}_{\mathcal{O}}$.