Quantified Positive Temporal Constraints[*]

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Abstract. A positive temporal template (or a positive temporal constraint language) is a relational structure whose relations can be defined over countable dense linear order without endpoints using a relational symbol $\leq$, logical conjunction and disjunction. This paper gives a complete complexity characterization for quantified constraint satisfaction problems (QCSP) over positive temporal languages. Although the constraint satisfaction problem (CSP) for an arbitrary positive temporal language is trivial (all these templates are closed under constant functions), the corresponding QCSP problems are decidable in LOGSPACE or complete for one of the following classes: NLOGSPACE, P, NP or PSPACE.

1 Introduction

Constraint Satisfaction Problems provide a uniform approach to research on a wide variety of combinatorial problems. Undisputedly, the most interesting problem in this area is to verify Dichotomy Conjecture posed by Feder and Vardi [1]. It says that every constraint satisfaction problem on a finite domain is either tractable or NP-complete and was inspired by Schaefer's Dichotomy Theorem for CSP on a two element set [2]. When algebraic approach came on the scene the works on dichotomy conjecture were sped up [3]. Although the main goal has not yet been attained, many interesting results were published and many interesting techniques were developed [4, 5]. Besides earlier results on constraints over infinite domains [6, 7], a new approach was quite recently proposed and developed by Manuel Bodirsky [8] and co-authors. This framework concentrates on relational structures that are ω-categorical. Many results concerning both CSP and QCSP [9] over finite domains were generalized to infinite ones. Moreover, new results were established. Among them there are full characterizations of complexity for both CSP and QCSP of equality constraint languages [10, 11].

Our paper is the next step in this research area. In general, we consider quantified constraint satisfaction problems for sets of relations definable over $\langle \mathbb{Q}, < \rangle$. In particular, we restrict ourselves to templates definable with $\wedge, \vee$ and $\leq$, i.e., we do not consider negation. We name such relations and languages positive temporal. As in [12], we refer to an arbitrary relation defined over $\langle \mathbb{Q}, < \rangle$ as a temporal relation.

Our main contribution is a complexity characterization of QCSP problems over positive temporal languages summarized in Theorem 1 below. We follow the algebraic approach to constraint satisfaction problems: we first classify positive temporal languages depending on their surjective polymorphisms and then give the complexity of QCSP for each obtained class.

Theorem 1 (The Main Theorem). *Let Γ be a language of positive temporal relations, then one of the following holds.*

1. *Each relation in Γ is definable by a conjunction of equations ($x_1 = x_2$) and then $QCSP(\Gamma)$ is decidable in LOGSPACE.*
2. *Each relation in Γ is definable by a conjunction of weak inequalities ($x_1 \leq x_2$). If there exists a relation in Γ that is not definable as a conjunction of equalities, then $QCSP(\Gamma)$ is NLOGSPACE-complete.*
3. *Each relation in Γ is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_1} \leq x_{i_2} \vee \ldots \vee x_{i_1} \leq x_{i_k})$ and then, provided Γ satisfies neither condition 1 nor 2, the set $QCSP(\Gamma)$ is P-complete.*
4. *Each relation in Γ is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_2} \leq x_{i_1} \vee \ldots \vee x_{i_k} \leq x_{i_1})$ and then provided Γ satisfies neither condition 1 nor 2, the set $QCSP(\Gamma)$ is P-complete.*
5. *Each relation in Γ is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_1} = y_{i_1} \vee \ldots \vee x_{i_k} = y_{i_k})$ and then provided Γ does not belong to any of the classes 1–4, the set $QCSP(\Gamma)$ is NP-complete.*
6. *The problem $QCSP(\Gamma)$ is PSPACE-complete.*

Related work. The complete characterization is quite complex and does not fit into one paper. Therefore some parts of Theorem 1 are proved in a companion paper [13]. In particular, we prove there that each QCSP problem over positive temporal relations is either in P or is NP-hard. Here we provide the complete characterization of the NP-hard case, distinguishing between NP-complete and PSPACE-complete cases (items 5 and 6 of Theorem 1). We also give complexity proofs for items 3 and 4 of Theorem 1, leaving the algebraic characterization in [13].

Quantified constraint satisfaction problems over temporal relations were investigated in [11, 14]. In particular, it is shown there that quantified problems from item 1 and 2 of Theorem 1 belong to LOGSPACE and NLOGSPACE, respectively. Our result substantially improves these results in the sense that we consider a strictly more expressive class of constraint languages. As in [11] we use the surjective preservation theorem.

The area of *CSP* may be often seen as a good framework for many problems in AI. In context of our characterization the well-motivated AND/OR precedence constraints [15] should be noted. They are closely related to languages from items 3 and 4. It might be said that we consider quantified positive variations of AND/OR precedence constraints.

In a very recent paper [12] the authors give a classification of *CSP* over temporal languages depending on their polymorphisms. Although it sounds similar, it is different from our classification. We deal with positive temporal languages and surjective polymorphisms, which are used to classify *QCSP* problems (as opposed to *CSP* problems considered in [12]). In the case of positive temporal languages, the classification based on polymorphisms is trivial: all these languages fall into the same class because they are all closed under constant functions — as a consequence all *CSP* problems for positive temporal languages are trivial. To obtain our classification we use methods different from those used in [12].

Outline of the paper. In Section 2, we give some preliminaries. Among others, we recall a definition of a surjective polymorphism and surjective preservation theorem, which is the most important tool in algebraic approach to *QCSP*. In [13] it is shown that the problem $QCSP(\Gamma)$ is NP-hard if and only if Γ has essentially unary surjective polymorphisms only. Sections 3 and 4 are devoted to classify positive temporal languages preserved by essentially unary surjections. In Section 3 we show that there are only five different classes of positive temporal relations with different surjective unary polymorphisms. If a positive temporal language Γ is closed under all unary surjective polymorphisms, then, as it was shown in [11], each relation of Γ may be defined as in item 5 of Theorem 1 and $QCSP(\Gamma)$ is NP-complete. If a positive temporal language is preserved by some non-trivial subset of unary surjections, then $QCSP(\Gamma)$ is PSPACE-complete. We prove it in Section 4. The last section contains a complexity proof for cases 3 and 4.

2 Preliminaries

Relational structures. In most cases we follow the notation from [8, 11]. We consider only relations defined over countable domains and hence whenever we write a domain or D we mean a countable set. Let τ be some relational (in this paper always finite) signature i.e., a set of relational symbols with assigned arity. Then Γ is a τ-structure over domain D if for each relational symbol R_i from τ, it contains a relation of according arity defined on D. In the rest of the paper we usually say relational language (or pattern) instead of relational structure. Moreover, we use the same notation for relational symbols and relations.

Automorphisms of Γ constitute a group with respect to composition. An orbit of a k-tuple t in Γ is the set of all tuples of the form $\langle \Pi(t_1), \ldots, \Pi(t_k) \rangle$ for all automorphisms Π. We say that a group of automorphisms of Γ is oligomorphic if for each k it has a finite number of orbits of k-tuples. Although there are many different ways of introducing a concept of ω-categorical structures we do it by the following theorem [16].

Theorem 2. *(Engeler, Ryll-Nardzewski, Svenonius) Let Γ be a relational structure. Then Γ is ω-categorical if and only if the automorphism group of Γ is oligomorphic.*

Polymorphisms. Let R be a relation of arity n defined over D. We say that a function $f : D^m \to D$ is a polymorphism of R if for all $a^1, \ldots, a^m \in R$ (where a^i, for $1 \le i \le m$, is a tuple $\langle a_1^i, \ldots, a_n^i \rangle$), we have $\langle f(a_1^1, \ldots, a_1^m), \ldots, f(a_n^1, \ldots, a_n^m) \rangle \in R$. Then we say that f preserves R or that R is closed under f. A polymorphism of Γ is a function that preserves all relations of Γ. By $Pol(\Gamma)$ we denote the set of polymorphisms of Γ, and by $sPol(\Gamma)$ — the set of surjective polymorphisms.

An operation f of arity n is *essentially unary* if there exists a unary operation f_0 such that $f(x_1, \ldots, x_n) = f_0(x_i)$ for some fixed $i \in \{1, \ldots, n\}$. An operation that is not essentially unary is called *essential*.

Quantified constraint satisfaction problems. Let Γ contain $R_1, \ldots, R_k$. Then a conjunctive positive formula (*cp-formula*) over Γ is a formula of the following form:

$$Q_1 x_1 \ldots Q_n x_n (R_1(\boldsymbol{v_1}) \wedge \ldots \wedge R_k(\boldsymbol{v_k})), \tag{1}$$

where $Q_i \in \{\forall, \exists\}$ and $\boldsymbol{v_j}$ are vectors of variables $x_1, \ldots, x_n$.

A *QCSP(Γ)* is a problem to decide whether a given cp-formula without free variables over the structure Γ is true or not. Note that by downward Löwenheim-Skolem Theorem we can focus on countable domains only.

If all quantifiers in (1) are existential then such a cp-formula is called positive primitive (*pp-formula*). A problem to decide whether a given pp-formula over Γ is satisfiable is well-known as a *constraint satisfaction problem*.

A relation R has a *cp-definition* in Γ if there exists a cp-formula $\phi(x_1, \ldots, x_n)$ over Γ such that for all $a_1, \ldots, a_n$ we have $R(a_1, \ldots, a_n)$ iff $\phi(a_1, \ldots, a_n)$ is true. The set of all relations cp-definable in Γ is denoted by $[\Gamma]$.

Lemma 1 ([11]). *Let Γ_1, Γ_2 be relational languages. If every relation in Γ_1 has a cp-definition in Γ_2, then QCSP(Γ_1) is log-space reducible to QCSP(Γ_2).*

The following results link $[\Gamma]$ with *sPol(Γ)*. The idea behind Theorem 3 is that the more Γ can express, in the sense of cp-definability, the less polymorphisms are contained in *sPol(Γ)*. Moreover, the converse is also true. This theorem is called *surjective preservation theorem*.

Theorem 3 ([11]). *Let Γ be an ω-categorical structure. Then a relation R has a cp-definition in Γ if and only if R is preserved by all surjective polymorphisms of Γ.*

As a direct consequence of Lemma 1 and Theorem 3 we obtain the following.

Corollary 1 ([11]). *Let Γ_1, Γ_2 be ω-categorical structures. If sPol(Γ_2) $\subseteq$ sPol(Γ_1), then QCSP(Γ_1) is log-space reducible to QCSP(Γ_2).*

Games and cp-definitions. Sometimes it is useful to see a cp-formula ψ without free variables as a two-player game. The game consists of alternating moves of existential and universal player. All variables are evaluated in the order they occur in the quantifier prefix, the existential player evaluates existentially quantified variables and the universal player evaluates universally quantified variables. At the end of the game, the players establish a valuation q from the variables of ψ into the set of rational numbers. We say that one variable is earlier (later) than the another one if it occurs earlier (later) in the quantifier prefix. If at the end of the game, the valuation q satisfies ψ, then the existential player wins; otherwise, the universal player is the winner. If the existential player has a winning strategy, then ψ is true; otherwise, if there exists a winning strategy for the universal player, then ψ is false.

Quantified Equality Constraints. Concerning patterns that allow equations and all logical connectives the following classification is known [11].

1. **Negative languages**. Relations of such a language are definable as CNF-formulas whose clauses are either equalities ($x = y$) or disjunctions of disequalities ($x_1 \neq y_1 \vee \ldots \vee x_k \neq y_k$). For each negative Γ the problem $QCSP(\Gamma)$ is contained in LOGSPACE.

2. **Positive languages**. Relations may be defined as a conjunction of disjunctions of equalities $(x_1 = y_1 \vee \ldots \vee x_k = y_k)$. For each positive Γ not being negative the problem $QCSP(\Gamma)$ is NP-complete.
3. In any other case the problem $QCSP(\Gamma)$ is PSPACE-complete.

Note that the class 1 from Theorem 1 is a subset of Negative languages and the class 5 is just the class of Positive languages.

To give our characterization we need the following result. It may be inferred from lemmas given in Section 7 in [11].

Lemma 2. *Let Γ be an equational positive constraint language that is preserved by an essential operation on D with infinite image. Then Γ is preserved by all operations, and Γ is negative.*

Corollary 2. *If an equational positive constraint language Γ is positive, but not negative, then $sPol(\Gamma)$ contains only essentially unary polymorphisms.*

Quantified Positive Temporal Constraints. Now, we focus on positive temporal relations announced in the introduction. All of them are defined over the set of rational numbers using a relational symbol $\leq$ and connectives $\wedge, \vee$. Therefore our results concerning positive temporal relations generalize those for positive equality languages. Since the only relational symbol we use is interpreted as a weak linear order over rational numbers, for each positive temporal structure Γ the set $sPol(\Gamma)$ contains all automorphisms that preserve order, i.e., all increasing unary surjections $f : \mathbb{Q} \to \mathbb{Q}$. Thus, using Theorem 2, it is not hard to see that all positive temporal languages are ω-categorical.

In [13] we show that each temporal relation is closed not only under all increasing functions but also under all weakly increasing surjections.

3 Surjective Unary Polymorphisms of Temporal Relations

This section examines positive temporal relations that are closed only under surjective unary polymorphisms. We want to divide this subset of positive temporal languages into classes each of which contains Γ_1 and Γ_2 if and only if $sPol(\Gamma_1) = sPol(\Gamma_2)$ (or equivalently $[\Gamma_1] = [\Gamma_2]$). Such a classification facilitates providing complexity results — see Theorem 3.

First we give some preliminary definitions. A permutation of a finite set is a bijection from this set to itself. Let $A = \{a_1, \ldots, a_n\}$ be a finite ordered set such that $a_1 < \ldots < a_n$. We say that a permutation π of A is a cycle of A if there exists $i \leq n$ such that $\pi(a_i) < \pi(a_{i+1}) < \ldots < \pi(a_n) < \pi(a_1) < \ldots < \pi(a_{i-1})$. Similarly, π is a reversed cycle if there exists $i \leq n$ such that $\pi(a_i) > \pi(a_{i+1}) > \ldots > \pi(a_n) > \pi(a_1) > \ldots > \pi(a_{i-1})$.

Definition 1. *We say that a relation R is closed under all permutations (respectively, under all cycles or reversed cycles) if for every tuple $\langle q_1, \ldots, q_n \rangle \in R$ and every permutation (respectively, every cycle or reversed cycle) π of the set $\{q_1, \ldots, q_n\}$ we have $\langle \pi(q_1), \ldots, \pi(q_n) \rangle \in R$.*

Note that in the definition above we permute the set $\{q_1, \ldots, q_n\}$ (and not the set of indices $\{1, \ldots, n\}$), which may have less then n elements if $q_1, \ldots, q_n$ are not pairwise distinct.

The preceding definitions concern closure under various kinds of permutations. Although they may look quite similar to closure under polymorphisms, they are different. Below we give some, important for us, examples of (unary) surjective polymorphisms of positive temporal relations. They are all of the type: $\mathbb{Q} \to \mathbb{Q}$.

Definition 2. *We say that a surjection $f : \mathbb{Q} \to \mathbb{Q}$ is weakly half-increasing (respectively weakly half-decreasing) if there exist two irrational real numbers x and y such that*

- *f restricted to the set $\{q \in \mathbb{Q} \mid q < x\}$ as well as f restricted to the set $\{q \in \mathbb{Q} \mid q > x\}$ is weakly increasing (respectively, weakly decreasing), and*
- *for all $q < x$ we have $f(q) > y$ (respectively $f(q) < y$) and for all $q > x$ we have $f(q) < y$ (respectively $f(q) > y$).*

A weakly half-increasing or weakly half-decreasing function is called weakly half-monotone.

Example 1. Recall that all countable dense linear orders without endpoints are isomorphic. In particular, $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$ are isomorphic, so we may identify $\mathbb{Q}$ with $\mathbb{Q} \setminus \{0\}$ and think of 0 as an irrational number in $\mathbb{Q} \setminus \{0\}$. Then the function $f : \mathbb{Q} \setminus \{0\} \to \mathbb{Q} \setminus \{0\}$ defined by $f(q) = \frac{1}{q}$ is weakly half-decreasing and the function defined by $f(q) = \frac{-1}{q}$ is weakly half-increasing.

The unary operation $- : \mathbb{Q} \to \mathbb{Q}$ is defined as $-(x) = -x$ in usual sense.

The rest of this section is devoted to prove the following result.

Theorem 4. *Let Γ be a set of positive temporal relations such that $sPol(\Gamma)$ contains only essentially unary functions. Then exactly one of the following cases holds.*

1. *$sPol(\Gamma)$ is the set of all unary surjections of $\mathbb{Q}$.*
2. *$sPol(\Gamma)$ is the set of all weakly increasing, weakly decreasing or weakly half-monotone surjections of $\mathbb{Q}$.*
3. *$sPol(\Gamma)$ is the set of all weakly increasing or weakly decreasing surjections of $\mathbb{Q}$.*
4. *$sPol(\Gamma)$ is the set of all weakly increasing or weakly half-increasing surjections of $\mathbb{Q}$.*
5. *$sPol(\Gamma)$ is the set of all weakly increasing surjections of $\mathbb{Q}$.*

A similar classification considering (not necessarily surjective) unary polymorphisms was obtained in [12]. Weakly half-increasing polymorphisms correspond in some way to the function cyc from that paper. In turn, positive temporal relations preserved by weakly half-decreasing functions correspond to temporal relations closed under $-$ and cyc.

As indicated in Theorem 4, there are only four interesting classes of unary polymorphisms of positive temporal relations: weakly increasing, weakly decreasing, weakly half-increasing, and weakly half-decreasing. The following lemmas say that if some positive temporal relation is closed under one polymorphism of a given class, then it is closed under all polymorphisms of this class.

Lemma 3. *If sPol(R) contains a weakly decreasing unary surjection f, then it contains all weakly decreasing unary surjections.*

Lemma 4. *If sPol(R) contains a weakly half-increasing unary surjection f, then it contains all weakly half-increasing unary surjections. If sPol(R) contains a weakly half-decreasing unary surjection f, then it contains all weakly decreasing, all weakly half-increasing and all weakly half-decreasing unary surjections.*

Now, we relate various surjective polymorphisms to closures under various kinds of permutations (see for example Definition 1). In particular, Lemma 5 below implies that the set of positive temporal relations closed under all permutations equals to the set of positive languages from [11].

Lemma 5. *A positive temporal relation R is closed under all permutations iff sPol(R) contains all unary surjections of $\mathbb{Q}$.*

Lemma 6. *A positive temporal relation R is closed under all cycles iff sPol(R) contains all weakly half-increasing surjections of $\mathbb{Q}$.*

Lemma 7. *A positive temporal relation R is closed under all reversed cycles iff sPol(R) contains all weakly half-decreasing surjections of $\mathbb{Q}$.*

Since we are interested in surjective functions, we can claim the following.

Lemma 8. *Let f be a unary, surjective operation on Q, then there exist:*

- *an infinite, strictly monotone sequence $(a_n)_{n\in\mathbb{N}}$ of rational numbers such that $\lim_{n\to\infty} f(a_n) = +\infty$*
- *an infinite, strictly monotone sequence $(b_n)_{n\in\mathbb{N}}$ of rational numbers such that $\lim_{n\to\infty} f(b_n) = -\infty$*

To prove Theorem 4, we show that if $sPol(\Gamma)$ contains any function that is neither weakly monotone nor weakly half-monotone, then it contains all unary rational functions or equivalently, by Lemma 5, is closed under all permutations.

Lemma 9. *Let R be a positive temporal relation such that $sPol(R)$ contains a function f that is neither weakly increasing nor weakly decreasing nor weakly half-monotone. Let $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ be two strictly monotone sequences satisfying the following: $\lim_{n\to\infty} f(c_n) = +\infty$ and $\lim_{n\to\infty} f(d_n) = -\infty$. Then R is closed under all permutations.*

Proof. (of Theorem 4) Suppose that $sPol(\Gamma)$ contains only essentially unary functions. If $sPol(\Gamma)$ contains a function f that is neither weakly monotone nor weakly half-monotone, then by Lemma 8 we find two strictly monotone sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f(a_n) = +\infty$ and $\lim_{n\to\infty} f(b_n) = -\infty$. Then by Lemma 9 every relation in Γ is closed under all permutations, so by Lemma 5 $sPol(R)$ contains all essentially unary surjections. Hence $sPol(\Gamma)$ is the set of all essentially unary surjections of $\mathbb{Q}$ and we are in case 1.

Now assume that $sPol(\Gamma)$ contains only weakly monotone or weakly half-monotone surjections. There are four cases, depending on whether $sPol(\Gamma)$ contains a weakly decreasing surjection or a weakly half-increasing surjection.

If $sPol(\Gamma)$ contains a weakly decreasing surjection and a weakly half-increasing surjection, then it contains a weakly half-decreasing surjection and by Lemmas 3 and 4, it contains all weakly decreasing and all weakly half-monotone surjections of $\mathbb{Q}$, so we are in case 2.

If $sPol(\Gamma)$ contains a weakly decreasing surjection and it does not contain any weakly half-increasing surjection, then by lemmas 3 and 4 it contains all weakly decreasing and it does not contain any weakly half-monotone surjections, so we are in case 3.

If $sPol(\Gamma)$ does not contain any weakly decreasing surjection and it contains a weakly half-increasing surjection, then by Lemma 4 it contains all weakly half-increasing and it does not contain weakly decreasing surjections, so we are in case 4.

Finally, if $sPol(\Gamma)$ does not contain any weakly decreasing surjection and it does not contain any weakly half-increasing surjection, then by Lemma 4 it does not contain any weakly decreasing nor weakly half-monotone surjection, so we are in case 5. □

Example 2. Recall from [11] that $(x_1 = x_2 \vee x_1 = x_3)$ is closed under all essentially unary surjections of $\mathbb{Q}$ — see also Section 2. Now, for each of the classes 2–5 of Theorem 4 we give representatives, that is, relations $R_{(5)}$–$R_{(2)}$ each of which belongs to exactly one of these classes.

The relation $R_{(5)}$ defined by $R_{(5)}(x_1, x_2, x_3) := (x_1 \leq x_2 \vee x_2 \leq x_3)$, as all positive temporal relations, is closed under all weakly increasing functions. Observe that $\langle 1, 2, 3 \rangle \in R_{(5)}$, but $\langle -1, -2, -3 \rangle \notin R_{(5)}$, so $R_{(5)}$ is not closed under weakly decreasing functions (and by Lemma 4 it is not closed under half-decreasing functions). Similarly, $\langle 1, 3, 2 \rangle \in R_{(5)}$, but $\langle 3, 2, 1 \rangle \notin R_{(5)}$, so $R_{(5)}$ is not closed under cycles (and thus it is not closed under weakly half-increasing functions). The relation $R_{(4)}$ defined by $R_{(4)}(x_1, x_2, x_3) := (x_1 \leq x_2 \vee x_2 \leq x_3) \wedge (x_2 \leq x_3 \vee x_3 \leq x_1) \wedge (x_3 \leq x_1 \vee x_1 \leq x_2)$ is a conjunction of the relations $(x_{\Pi(1)} \leq x_{\Pi(2)} \vee x_{\Pi(2)} \leq x_{\Pi(3)})$ where Π ranges over all cycles of the set $\{1, 2, 3\}$, so it is closed under all cycles. Since $\langle 1, 2, 3 \rangle \in R_{(4)}$ and $\langle 3, 2, 1 \rangle \notin R_{(4)}$, it is not closed under weakly decreasing or weakly half-decreasing functions. It is easy to observe that the relation $R_{(3)}$ defined by $R_{(3)}(x_1, x_2, x_3) := (x_1 \leq x_2 \vee x_2 \leq x_3) \wedge (x_3 \leq x_2 \vee x_2 \leq x_1)$ is closed under weakly decreasing functions. Since $\langle 2, 1, 3 \rangle \in R_{(3)}$ and $\langle 3, 2, 1 \rangle \notin R_{(3)}$, this relation is not closed under cycles and by Lemmas 6 and 4 it is not closed under any weakly half-monotone surjection. Let a relation $R_{(2)}$ be defined as a conjunction of the clauses $(x_{\Pi(1)} \leq x_{\Pi(2)} \vee x_{\Pi(2)} \leq x_{\Pi(3)} \vee x_{\Pi(3)} \leq x_{\Pi(4)})$ where Π ranges over all cycles and reversed cycles of the set $\{1, 2, 3, 4\}$, so it obviously must be closed under all cycles and reversed cycles. Note that cycles and reversed cycles are 8 out of total 24 permutations of the set $\{1, 2, 3, 4\}$ (this explains why we could not use a ternary relation as an example here — all permutations of the set $\{1, 2, 3\}$ are either cycles or reversed cycles). To see that $R_{(2)}$ is not closed under all permutations observe that $\langle 4, 3, 2, 1 \rangle \notin R_{(2)}$, but $\langle 2, 1, 3, 4 \rangle \in R_{(2)}$.

Finally, we show that all these relations $(R_{(2)}$–$R_{(5)})$ are closed under essentially unary surjections only. Let $R(x_1, \ldots, x_k)$ where $k = 3, 4$ be one of these relations.

Then a relation $\bigwedge_{\Pi \in S_k} R(x_{\Pi(1)}, \ldots, x_{\Pi(k)})$ where S_k is a set of all permutations on k elements is equivalent to a relation R' defined by $\bigvee_{i \neq j} x_i = x_j$. Because R' is positive and non-negative, by Corollary 2 and Theorem 3, we have that R is closed under unary surjections only.

4 PSPACE-complete Positive Temporal Languages

Recall from Section 2 the complexity characterization of equational languages. By corollaries 2 and 1, the problem $QCSP(\Gamma)$ where Γ is closed under essentially unary functions only is NP-hard. Likewise we know that $QCSP$ for languages from item 1 of Theorem 4 is NP-complete. This section is devoted to show PSPACE-completeness for $QCSP$ of languages with surjective polymorphisms from items 2–5 of Theorem 4.

Membership in PSPACE is the simpler part of the proof and is common for all, not only positive, temporal relations.

Proposition 1. *For every temporal language Γ, the problem $QCSP(\Gamma)$ is decidable in PSPACE.*
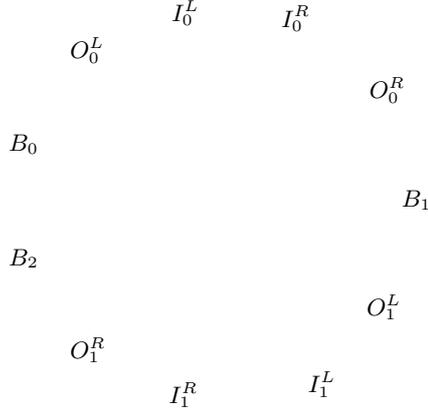
In the rest of the section we prove hardness. Note that the set of surjective polymorphisms from item 2 contains sets of surjective polymorphisms from each of items 2–5. Therefore, by Theorem 3 and Corollary 1, it is enough to show PSPACE-hardness of $QCSP$ for positive temporal languages closed under all weakly monotone and all weakly half-monotone surjections only.

Theorem 5. *Let Γ be a set of positive languages closed only under essentially unary functions. If $sPol(\Gamma)$ is the set of all weakly increasing, weakly decreasing and weakly half-monotone surjections of $\mathbb{Q}$, then $QCSP(\Gamma)$ is PSPACE-hard.*

Because of Theorem 3 and Corollary 1, it is enough to choose just one language with appropriate set of surjective polymorphisms and show PSPACE hardness for this language. Our choice is the language Γ_{Circle} defined below. We show that it is closed only under all weakly increasing, weakly decreasing and weakly half-monotone surjections of $\mathbb{Q}$. In fact, it is enough to show that Γ_{circle} is closed only under unary surjections of $\mathbb{Q}$ and that is closed under all reversed cycles – see lemmas 4 and 7. Finally, we show that $QCSP(\Gamma_{Circle})$ is PSPACE-hard and in consequence we prove Theorem 5.

Definition of Γ_{Circle}. First we present some auxiliary relations that shorten the definition. Let $\boldsymbol{v_A}$ be a vector $\langle B_0, O_0^L, I_0^L, I_0^R, O_0^R, B_1, O_1^L, I_1^L, I_1^R, O_1^R, B_2 \rangle$ of variables ranging over $\mathbb{Q}$. The corresponding set of variables is denoted by Var_A. In the following we call Var_A the set of arena variables. We sometimes see a vector $\boldsymbol{v_A}$ as a function from $\{0, \ldots, 10\}$ to Var_A.

Let $Arenas$ be a set of vectors $\Pi(\boldsymbol{v_A})$ for all cycles and reversed-cycles Π of the set $\{0, \ldots, 10\}$. Note that the set $Arenas$ may be represented in some way using Fig. 1. To obtain one of linear orders that is represented by this circle, we have just to tear it apart and orientate. If we orientate it clockwise, then we represent some $\Pi_C(\boldsymbol{v_A})$ where Π_C is a cycle. Otherwise, if we orientate it anticlockwise, then we represent some $\Pi_{RC}(\boldsymbol{v_A})$ where Π_{RC} is a reversed cycle.

$$I_0^L \qquad I_0^R$$
$$O_0^L$$
$$O_0^R$$
$$B_0$$
$$B_1$$
$$B_2$$
$$O_1^L$$
$$O_1^R$$
$$I_1^R \qquad I_1^L$$

Fig. 1. The representation of the set $Arenas$

Now, for each $\boldsymbol{v} \in Arenas$ we define a relation $Prefix_{\boldsymbol{v}} := \neg(y_0 < \ldots < y_{10})$ where $\boldsymbol{v} = \langle y_0, \ldots, y_{10} \rangle$. At this point we probably owe the reader one more explanation. Sometimes, when we think it is intuitive, we use negation as well as implication in the definition of relations. Nevertheless, they should be treated just as notational shortcuts and all relations we claim to be positive temporal are indeed definable by conjunction, disjunction, and $\leq$. In particular, $Prefix_{\boldsymbol{v}}$ may be defined as $\bigvee_{i=0}^{9} y_i \geq y_{i+1}$. Nevertheless, the situation where $Prefix_{\boldsymbol{v}}$ is falsified is more important for us. Then the arena variables are arranged in some linear order represented by a circle in Fig. 1.

In general, our intention is to model (see Definition 3) a boolean relation. Arena variables set in some order presented in Fig. 1 constitute some kind of arena. When some other variable is set strictly between O_0^L and O_0^R then we see its value as a boolean zero, and if some variable is set strictly between O_1^L and O_1^R then we see its value as a boolean one. We need also I_0^L, I_0^R, and I_1^L, I_1^R. Sometimes we want to say: 'If a variable x is equal to zero, then a variable y is also equal to zero'. Unfortunately, concerning positive temporal relations we are unable to write something like $(O_0^L < x < O_0^R) \rightarrow (O_0^L < y < O_0^R)$. Instead we write $(O_0^L < x < O_0^R) \rightarrow (I_0^L \leq y \leq I_0^R)$ and assure that I_0^L, I_0^R are always strictly between O_0^L, O_0^R. Similarly, we assure that I_1^L, I_1^R are always strictly between O_1^L, O_1^R. This is the general idea, but sometimes because of technical reasons we also use O_0^{L1}, O_0^{L2} etc.

Concerning positive temporal relations closed under reversed cycles it is hard to say that some variable must be set on the left (or on the right) of the another variable. Far more natural is to say that some variable is inside the interval set by values of other variables or outside such an interval. We define $In_{\boldsymbol{v}}(x, y_1, y_2)$ equal to $((y_1 < y_2) \rightarrow (y_1 \leq x \leq y_2)) \wedge ((y_2 < y_1) \rightarrow (x \leq y_2 \vee x \geq y_1))$ if $\boldsymbol{v} = \Pi_C(\boldsymbol{v_A})$ for some cycle Π_C; and equal to $((y_1 < y_2) \rightarrow (x \leq y_1 \vee x \geq y_2)) \wedge ((y_2 < y_1) \rightarrow (y_2 \leq x \leq y_1))$ if $\boldsymbol{v} = \Pi_{RC}(\boldsymbol{v_A})$ for some reversed-cycle Π_{RC}. Similarly, we define $Out_{\boldsymbol{v}}(x, y_1, y_2)$ equal to $((y_1 < y_2) \rightarrow (x \leq y_1 \vee x \geq y_2)) \wedge ((y_2 < y_1) \rightarrow (y_2 \leq x \leq y_1))$ if $\boldsymbol{v} = \Pi_C(\boldsymbol{v_A})$ for some cycle Π_C; and equal to $((y_1 < y_2) \rightarrow (y_1 \leq x \leq y_2)) \wedge ((y_2 < y_1) \rightarrow (x \leq y_2 \vee x \geq y_1))$ if $\boldsymbol{v} = \Pi_{RC}(\boldsymbol{v_A})$ for some reversed-cycle Π_{RC}.

Example 3. For every $\boldsymbol{v} \in Arenas$ the following formulas are always true:

1. $(\neg Prefix_{\boldsymbol{v}}) \rightarrow In_{\boldsymbol{v}}(I_0^R, O_0^R, O_0^L)$
2. $(\neg Prefix_{\boldsymbol{v}}) \rightarrow Out_{\boldsymbol{v}}(O_0^R, I_0^R, I_0^L)$

The positive temporal language Γ_{Circle} consists of three relations: *UImp*, *BImp*, and *Final*. Each relation $R \in \Gamma_{Circle}$ is of the form $\bigwedge_{\boldsymbol{v} \in Arenas} \phi_{\boldsymbol{v}}^R$. By using this conjunction we assure that R is closed under all cycles and reversed cycles.

1. First of our relations is *UImp*$(\boldsymbol{v_A}, p, O^L, O^R, f, I^L, I^R)$ with

$$\phi_{\boldsymbol{v}}^{UImp} := Prefix_{\boldsymbol{v}} \vee Out_{\boldsymbol{v}}(p, O^L, O^R) \vee In_{\boldsymbol{v}}(f, I^L, I^R). \tag{2}$$

The name *UImp* stands for unary implication. It is justified by the context in which we use it. If both $Prefix_{\boldsymbol{v}}$ and $Out_{\boldsymbol{v}}(p, O^L, O^R)$ are falsified, then $In_{\boldsymbol{v}}(f, I^L, I^R)$ must be satisfied. We use this relation to express the implication: 'if v represents different values in an appropriate order and p is a value in the interval from I^L to I^R, then f also is a value in this interval'.

2. We have also binary implication *BImp*$(\boldsymbol{v_A}, p_1, p_2, O^L, O^R, f, I^L, I^R)$ with

$$\phi_{\boldsymbol{v}}^{BImp} := Prefix_{\boldsymbol{v}} \vee Out_{\boldsymbol{v}}(p_1, O^L, O^R) \vee Out_{\boldsymbol{v}}(p_2, O^L, O^R) \vee In_{\boldsymbol{v}}(f, I^L, I^R). \tag{3}$$

If $Prefix_{\boldsymbol{v}}$ as well as $Out_{\boldsymbol{v}}(p_1, O^L, O^R)$ and $Out_{\boldsymbol{v}}(p_2, O^L, O^R)$ are falsified, then $In_{\boldsymbol{v}}(f, I^L, I^R)$ must be satisfied.

3. Finally there is *Final*$(\boldsymbol{v_A}, f_0, f_1)$ with

$$\phi_{\boldsymbol{v}}^{Final} := Prefix_{\boldsymbol{v}} \vee Out_{\boldsymbol{v}}(f_0, B_0, B_2) \vee Out_{\boldsymbol{v}}(f_1, B_0, B_2). \tag{4}$$

We want to see it in the following way. If $Prefix_{\boldsymbol{v}}$ is falsified, then $Out_{\boldsymbol{v}}(f_0, B_0, B_2)$ or $Out_{\boldsymbol{v}}(f_1, B_0, B_2)$ must be satisfied.

Lemma 10. *The positive temporal language Γ_{Circle} is closed under weakly increasing, weakly decreasing, and weakly half monotone surjections only.*

PSPACE-hardness of QCSP(Γ_{Circle}). The hardness proof for *QCSP(Γ_{Circle})* is based on the proof of PSPACE-hardness of $QCSP(x_1 \neq x_2 \vee x_1 = x_3)$ from [11]. We define analogous notions and follow analogous reasoning.

Definition 3. *A relation $R \subseteq \{0,1\}^n$ is force definable if there exists a prenex formula*

$$\Phi_{R,f_0,f_1}(\boldsymbol{v_A}, O_0^{L1}, O_0^{R1}, O_1^{L1}, O_1^{R1}, x_1, \ldots, x_n) = \mathcal{Q}\phi$$

over Γ_{Circle} that satisfies all of the following.

1. *$\mathcal{Q}$ is a quantifier prefix and ϕ is a quantifier-free part.*
2. *The quantifier prefix $\mathcal{Q}$ contains f_0 and f_1 as its two last variables, and they are both existentially quantified.*
3. *The set of free variables is equal to $\{\boldsymbol{v_A}, O_0^{L1}, O_0^{R1}, O_1^{L1}, O_1^{R1}, x_1, \ldots, x_n\}$.*
4. *Let $\boldsymbol{t} \in \{0,1\}^n$ and let $\boldsymbol{v} \in Arenas$. Let variables from Var_A be set to satisfy $\neg(Prefix_{\boldsymbol{v}})$ and let variables $O_0^{L1}, O_0^{R1}, O_1^{L1}, O_1^{R1}$ be set to satisfy*

- $In_{\boldsymbol{v}}(O_0^{L1}, B_0, O_0^L)$,
- $In_{\boldsymbol{v}}(O_0^{R1}, O_0^R, B_1)$,
- $In_{\boldsymbol{v}}(O_1^{L1}, B_1, O_1^L)$, *and*
- $In_{\boldsymbol{v}}(O_1^{R1}, O_1^R, B_2)$.

Further, let x_k for $k \in \{1, \ldots, n\}$ are set to satisfy $In_{\boldsymbol{v}}(x_k, I_i^L, I_i^R)$ iff $\boldsymbol{t}_k = i$ for $i = 0, 1$. Then the sentence $\Phi' := \mathcal{Q}(\phi \wedge \neg(In_{\boldsymbol{v}}(f_0, I_0^L, I_0^R) \wedge In_{\boldsymbol{v}}(f_1, I_1^L, I_1^R)))$ is false iff $\boldsymbol{t} \in R$.

5. *If values of arena variables satisfy $Prefix_{\boldsymbol{v}}$ for all $\boldsymbol{v} \in Arenas$; or, in case $Prefix_{\boldsymbol{v}}$ is falsified for some $\boldsymbol{v} \in Arenas$, free variables $O_0^{L1}, O_0^{R1}, O_1^{L1}, O_1^{R1}$ are set to satisfy*
   - $In_{\boldsymbol{v}}(O_0^{L1}, I_0^L, I_0^R) \wedge In_{\boldsymbol{v}}(O_0^{R1}, I_0^L, I_0^R) \vee$
   - $In_{\boldsymbol{v}}(O_1^{L1}, I_1^L, I_1^R) \wedge In_{\boldsymbol{v}}(O_1^{R1}, I_1^L, I_1^R)$;

   then Φ' is always true.

6. *(monotonicity) For any setting to the free variables of Φ_{R, f_0, f_1}, if the formula Φ' is true, then changing the value of any variable x_i to satisfy $(Out_{\boldsymbol{v}}(x_i, O_0^{L1}, O_0^{R1}) \vee Out_{\boldsymbol{v}}(x_i, O_1^{L1}, O_1^{R1}))$ preserves the truth of Φ'.*

As it was described in Section 2 we can see a sentence as a two-player game. The intuition behind Definition 3 is as follows. If free variables of Φ_{R, f_0, f_1} are set according to conditions from item 4 and $\boldsymbol{t} \in R$, then the universal player has a strategy to force the existential player to satisfy $In_{\boldsymbol{v}}(f_0, I_0^L, I_0^R)$ and $In_{\boldsymbol{v}}(f_1, I_1^L, I_1^R)$ where $\boldsymbol{v} \in Arenas$ and $Prefix_{\boldsymbol{v}}$ is falsified. But if $Prefix_{\boldsymbol{v}}$ is falsified for some $\boldsymbol{v} \in Arenas$ and the condition from item 5 is fulfilled, then the existential player is able to falsify $In_{\boldsymbol{v}}(f_0, I_0^L, I_0^R)$ or $In_{\boldsymbol{v}}(f_1, I_1^L, I_1^R)$.

Note that variables $O_0^{L1}, O_0^{R1}, O_1^{L1}, O_1^{R1}$ are different from $O_0^L, O_0^R, O_1^L, O_1^R$.

Lemma 11. *There exists a polynomial-time algorithm that, given a boolean circuit C as input, produces a force definition of the relation R_C containing, as tuples, exactly the satisfying assignments of the circuit.*

Similarly as in [11], we reduce from succinct graph unreachability. In this problem, the input is a boolean circuit with $2c$ inputs that represent a graph G whose vertices are the tuples in $\{0, 1\}^c$. There is a directed edge $(\boldsymbol{X}, \boldsymbol{Y})$ in the graph iff the circuit returns true given the input $(\boldsymbol{X}, \boldsymbol{Y})$. The question is to decide whether or not there is a directed path from $\boldsymbol{S}$ to $\boldsymbol{T}$. This problem is known to be *PSPACE*-complete.

Define $R_i \subseteq \{0, 1\}^{2c}$ to be the relation containing exactly the tuples $(\boldsymbol{X}, \boldsymbol{Y})$ such that there exists a directed path in G from $\boldsymbol{X}$ to $\boldsymbol{Y}$ of length less than or equal to 2^i. Then there is a path in G from $\boldsymbol{S}$ to $\boldsymbol{T}$ iff $(\boldsymbol{S}, \boldsymbol{T}) \in R_c$.

From Lemma 11 it is not hard to infer that R_0 is computable in polynomial time. Now, by induction we show that R_c is also computable by a polynomial algorithm.

Lemma 12. *The force definition of the relation R_c is computable in polynomial time.*

Proof. (of Theorem 5) Let $\Phi_{R_c, g_0, g_1}(\boldsymbol{v_A}, O_0^L, O_0^R, O_1^L, O_1^R, \boldsymbol{x}, \boldsymbol{y}) = Q_c \phi_c$ be a force definition of R_c. We use it now to give an instance of $QCSP(\Gamma_{Circle})$ that is true if and only if there is no path from $\boldsymbol{S}$ to $\boldsymbol{T}$ in the succinctly represented graph.

The instance created is

$$\forall \boldsymbol{v_A} Q_c \phi_c \wedge \boldsymbol{x} = \boldsymbol{s} \wedge \boldsymbol{y} = \boldsymbol{t} \wedge Final(\boldsymbol{v_A}, g_0, g_1)$$

where $\boldsymbol{s}_i = \boldsymbol{t}_j = I_k^L$ if $\boldsymbol{S}_i = \boldsymbol{T}_j = k$ for all $1 \leq i, j \leq c$ and $k = 0, 1$.

The universal player starts the game. To have a chance to win (to falsify) the sentence he must set arena variables to falsify $Prefix_{\boldsymbol{v}}$ for some $\boldsymbol{v} \in Arenas$. Otherwise each clause from ϕ_c is satisfied already at the beginning. If there is a path from $\boldsymbol{s}$ to $\boldsymbol{t}$, then the universal player can enforce the existential player to satisfy $In_{\boldsymbol{v}}(g_0, I_0^L, I_0^R)$ and $In_{\boldsymbol{v}}(g_1, I_1^L, I_1^R)$. It is not hard to verify that it contradicts $Final(\boldsymbol{v_A}, g_0, g_1)$ — see (4). If $(\boldsymbol{s}, \boldsymbol{t}) \notin R_c$, then the existential player can satisfy $Out_{\boldsymbol{v}}(g_0, B_0, B_2)$ or $Out_{\boldsymbol{v}}(g_1, B_0, B_2)$; and in consequence satisfy $Final(\boldsymbol{v_A}, g_0, g_1)$. □

5 PTIME-complete Positive Temporal Languages

This section is devoted to give the complexity proof for classes from items 3 and 4 of Theorem 1. We focus here of the former case, the latter one is dual and hence the whole reasoning is similar in both cases.

Let R_{Left}^k be equal to $(x_1 \leq x_2 \vee \ldots \vee x_1 \leq x_k)$, and Γ_{Left} be the set of relations R_{Left}^k for each natural number $k \geq 2$. In [13] it is shown that $QCSP(\Gamma_{Left})$ is log-space equivalent to $QCSP(\Gamma)$ where Γ is the language from case 3 of Theorem 1. Furthermore, we have that $QCSP(x_1 \leq x_2 \vee x_1 \leq x_2)$ is log-space equivalent to $QCSP(\Gamma_{Left})$. Observe that the definition of each R_{Left}^k have a simple tree-like structure where x_1 is a root and $x_2, \ldots, x_k$ are sons of x_1. Moreover, to denote a root of a clause C we write $root(C)$ and to denote a set of sons – $sons(C)$.

In this section whenever we write: a formula, we think of a cp-formula over Γ_{Left}.

Lemma 13. *Let ϕ be a formula defining a positive temporal relation. Assume that ϕ contains clauses $C_1 := (y \leq x_1 \vee \ldots \vee y \leq x_k)$ and $C_2 := (x_1 \leq z_1 \vee \ldots \vee x_1 \leq z_l)$. Then ϕ and ϕ' given by $\phi \wedge C_3$, where $C_3 := (y \leq z_1 \vee \ldots \vee y \leq z_l \vee y \leq x_2 \vee \ldots \vee y \leq x_k)$, are equivalent, that is, they define the same relation.*

In the following, we sometimes refer to a quantifier-free formula ϕ as to a set of clauses. We say that a set of clauses ϕ is $TClosed$ if for all pairs of clauses of the form $(y \leq x_1 \vee \ldots \vee y \leq x_k)$ and $(x_1 \leq z_1 \vee \ldots \vee x_1 \leq z_l)$, the clause $(y \leq z_1 \vee \ldots \vee y \leq z_l \vee y \leq x_2 \vee \ldots y\vee \leq x_k)$ also belongs to ϕ. By $TClosure(\phi)$ we denote the least $TClosed$ superset of ϕ.

By a simple induction, from Lemma 13 we can obtain the following.

Corollary 3. *Formulas ϕ and $TClosure(\phi)$ are equivalent.*

We show that the universal player has a winning strategy if and only if $TClosure(\phi)$ contains a clause of the form $(y \leq x_1 \vee \ldots \vee y \leq x_k)$ such that for each disjunct $y \leq x_i$ where $1 \leq i \leq k$ we have that either y or x_i is later and universal. We call such a clause *ultimate*.

Lemma 14. *Let ψ be a sentence and let Q be its quantifier prefix and ϕ its quantifier-free part. Then ψ is false if and only if $TClosure(\phi)$ contains an ultimate clause.*

To show the exact complexity of case 3 of Theorem 1 we use the emptiness problem for context-free grammars. It is well known that this problem is P-complete. We assume that the reader is familiar with the notion of the context-free grammar. By $\mathcal{L}(G)$ we denote a language generated by a context free-grammar $G = \langle N, \Sigma, R, S \rangle$.

Theorem 6. *Let Γ be a positive temporal language such that each of its relation is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_1} \leq x_{i_2} \vee \ldots \vee x_{i_1} \leq x_{i_k})$ and it is neither definable as a conjunction of equalities nor as a conjunction of inequalities. Then the problem $QCSP(\Gamma)$ is P-complete*

Proof. (About Membership) To obtain the result we give a logspace reduction from the problem $QCSP(x_1 \leq x_2 \vee x_1 \leq x_3)$ to the emptiness problem for context-free grammars. Let ψ be an instance of $QCSP(x_1 \leq x_2 \vee x_1 \leq x_3)$ with a quantifier prefix Q and a quantifier free-part ϕ. We construct a context-free grammar G_ψ such that $\mathcal{L}(G_\psi) \neq \emptyset$ if and only if ψ is false. By Lemma 14, it is enough to show G_ψ that generates a non-empty language if and only if $TClosure(\phi)$ contains an ultimate clause.

The reduction runs as follows. For each variable x of ψ, we show a grammar G_ψ^x that generates an empty language if and only if there is no ultimate clause C in $TClosure(\phi)$ with x being a root of C. Further, by G_ψ we take a grammar such that $\mathcal{L}(G_\psi) = \bigcup_{x \in Var(\psi)} \mathcal{L}(G_\psi^x)$ where $Var(\psi)$ is a set of all variables of ψ. Recall that the set of context-free grammars is closed under union and note that $\|G_\psi\| \leq c * \Sigma_{x \in Var(\psi)} \left\| G_\psi^x \right\|$ for some constant c.

We now turn to the definition of $G_x = \langle N_x, \Sigma_x, R_x, A_x \rangle$. For each variable y of ψ we introduce a nonterminal A_y. The set Σ_x contains a terminal symbol a_y for each variable y that is universal and later than x. If x is universal, then there is also a terminal a_y for each variable y that is earlier than x. Further, for each clause of the form $(x_1 \leq x_2 \vee x_1 \leq x_3)$ we have a rule $A_{x_1} \to A_{x_2} A_{x_3}$. For each terminal symbol a_y in Σ_x there is also a rule $A_y \to a_y$. It is clear, that such a reduction may be provided using logarithmic space.

Now, if $\mathcal{L}(G_\psi^x)$ contains a word $a_{x_1} \ldots a_{x_k}$, then, by a simple induction, we can show that a clause $(x \leq x_1 \vee \ldots \vee x \leq x_k)$ belongs to $TClosure(\phi)$. Since each x_i for $1 \leq i \leq k$ is universal and later than x or provided x is universal, earlier than x; this clause is ultimate. Similarly, if any ultimate clause $(x \leq x_1 \vee \ldots \vee x \leq x_k)$ belongs to $TClosure(\phi)$, then we can construct a parse tree that witnesses $a_{x_1} \ldots a_{x_k} \in \mathcal{L}(G_\psi^x)$

(About Hardness) The hardness proof is quite similar. This time we give a logspace reduction from the emptiness problem to the problem $QCSP(x_1 \leq x_2 \vee x_1 \leq x_3)$. $\square$

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