On Retracts of Algebras with Iteration

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Abstract. We show that iteration-congruent retracts of (completely) iterative algebras are (complete) Elgot algebras. Conversely, for an iteratable endofunctor H, every (complete) Elgot H-algebra arises as an iteration-congruent retract of a (completely) iterative H-algebra.

In a recent work, Goncharov *et al.* [5] study a relationship between different kinds of monads with iteration. In particular, they show that iteration-congruent retracts of completely iterative monads [1] yield complete Elgot monads [3]. Conversely, provided certain final coalgebras exist, every complete Elgot monad arises this way, that is, as an iteration-congruent retract of a completely iterative monad. In this note, we present similar results for algebras with iteration: (complete) Elgot algebras [2] and (completely) iterative algebras [1,6].

Let H be an endofunctor on a category $\mathscr{C}$ with binary coproducts. An *H-algebra with iteration* is a triple $\langle A, a : HA \to A, (\text{-})^\dagger \rangle$, where the $(\text{-})^\dagger$ operator assigns to every morphism $e : X \to A + HX$ a *solution*, that is, a morphism $e^\dagger : X \to A$ such that $e^\dagger = [\mathsf{id}, a] \cdot (\mathsf{id} + He^\dagger) \cdot e$. A *complete Elgot algebra* is an algebra with iteration in which the $(\text{-})^\dagger$ operator satisfies two additional axioms: *functoriality* and *compositionality* (see [2]). In what follows, given morphisms $e : X \to A + HX$ and $f : A \to B$, we write $f \bullet e$ for the morphism $(f + \mathsf{id}) \cdot e : X \to B + HX$.

Definition 1. *Let $\langle A, a, (\text{-})^\dagger \rangle$ be an H-algebra with iteration, and $\langle B, b \rangle$ be an H-algebra. We call a morphism $\rho : A \to B$ an* iteration-congruent retraction *if the following hold:*

1. *ρ is an algebra homomorphism $\langle A, a \rangle \to \langle B, b \rangle$,*
2. *ρ as a morphism in $\mathscr{C}$ has a section $\sigma : B \to A$,*
3. *ρ is iteration-congruent, that is, for all $e, f : X \to A + HX$, it is the case that $\rho \bullet e = \rho \bullet f$ implies $\rho \cdot e^\dagger = \rho \cdot f^\dagger$.*

Theorem 2. *Let $\langle A, a : HA \to A, (\text{-})^\dagger \rangle$ be a complete Elgot H-algebra, and $\langle B, b \rangle$ be an H-algebra. Then, given an iteration-congruent retraction $\rho : A \to B$, the algebra $\langle B, b \rangle$ can be given a complete Elgot structure with the solution of a morphism $e : X \to B + HX$ given as $e^\ddagger = \rho \cdot (\sigma \bullet e)^\dagger$. Moreover, in such a case ρ preserves solutions, that is, $(\rho \bullet e)^\ddagger = \rho \cdot e^\dagger$.*

A *completely iterative algebra* is an algebra $\langle A, a : HA \to A \rangle$ such that for a morphism $e : X \to A + HX$, there exists a unique solution $e^\dagger : X \to A$. Every completely iterative algebra, understood as an algebra with iteration, is a complete Elgot algebra. Thus, we obtain the following corollary of Theorem 2:

Corollary 3. *An iteration-congruent retract of a completely iterative algebra is a complete Elgot algebra.*

We also show that the converse holds if we assume an additional property of the endofunctor H. We say that H is *iteratable* [1] if the endofunctor $A + H(\text{-})$ has a final coalgebra for every object A. We write $H^\infty A$ to denote the carrier of such a final coalgebra. Importantly, if H is iteratable, each object A generates a free complete Elgot algebra $\mathbf{F}A = \langle H^\infty A, \tau, (\text{-})^\dagger \rangle$, which happens to be completely iterative (see [2] for a detailed description of these results).

Theorem 4. *If H is iteratable, then every complete Elgot H-algebra $\langle A, a, (\text{-})^\ddagger \rangle$ arises as an iteration-congruent retract of a completely iterative algebra. The retraction is given by the unique morphism from $\mathbf{F}A$, given as $\mathsf{out}^\ddagger : H^\infty A \to A$, where $\mathsf{out} : H^\infty A \to A + HH^\infty A$ is the action of the final coalgebra.*

An instance of such an iteration-congruent retraction can be found in Example 3.10 in [2]. Consider a complete lattice with a carrier A. Given a possibly infinite binary tree with labels from A in the leaves (that is, the carrier of the free completely iterative algebra of the endofunctor $X \mapsto X \times X$ on **Set** generated by A), the iteration-congruent retraction takes the join of all the leaves in the tree. This gives us a complete Elgot structure on the complete lattice A.

Adámek *et al.* [1,2] consider also non-complete versions of Elgot algebras and iterative algebras. For those, we assume that $\mathscr{C}$ is locally finitely presentable, and we require algebras with iteration to have solutions for morphisms $e : X \to A + HX$ if X is finitely presentable. The results shown in this note trivially hold in the non-complete version as well, since they do not rely on completeness and the construction of solutions does not require solving morphisms with different X's.

Theorem 4 is related to previous work [4] as follows. By [4, Theorem 5.7], the category of complete Elgot algebras is isomorphic to the category of (Eilenberg-Moore) H^∞-algebras, and so the retraction $H^\infty A \to A$ in question can be alternatively obtained as the H^∞-algebra structure on A. Conditions of Definition 1 are easily seen to be satisfied, e.g. (3) is due to the fact that any H^∞-algebra structure is always an H^∞-algebra morphism, and those isomorphically correspond to complete Elgot algebra morphisms.

References

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A Proofs

Full definition of complete Elgot algebras

For convenience of the reader, we recall the remaining parts of the definition of complete Elgot algebras, to be found in [2]. For two morhpisms $e : X \to Y + HX$ and $f : Y \to A + HY$, we form a morphism $f \mathbin{\blacksquare} e : Y + X \to A + H(Y + X)$ as $f \mathbin{\blacksquare} e = (\mathsf{id} + [H\mathsf{inl}, H\mathsf{inr}]) \cdot (f + \mathsf{id}) \cdot [\mathsf{inl}, e]$. The two omitted axioms are as follows:

- *Functoriality:* Let $e = X \to A + HX$ and $f = Y \to A + HY$ be morphisms, and $h : X \to Y$ be a coalgebra homomorphism, that is, $(\mathsf{id} + Hh) \cdot e = f \cdot h$. Then, $e^\dagger = f^\dagger \cdot h$.
- *Compositionality:* Let $e : X \to Y + HX$ and $f : Y \to A + HY$. Then, $(f^\dagger \bullet e)^\dagger = (f \mathbin{\blacksquare} e)^\dagger \cdot \mathsf{inr}$.

Proof of Theorem 2

We proceed with a number of facts:

(A) For all e it is the case that $\rho \cdot e^\dagger = \rho \cdot ((\sigma \cdot \rho) \bullet e)^\dagger$:

$$\begin{aligned} & \rho \bullet e = (\rho \cdot \sigma \cdot \rho) \bullet e = \rho \bullet ((\sigma \cdot \rho) \bullet e) && \text{(section-retraction, props. of } \bullet\text{)} \\ \Longrightarrow\ & \rho \cdot e^\dagger = \rho \cdot ((\sigma \cdot \rho) \bullet e)^\dagger && \text{(congruence)} \end{aligned}$$

(B) $(\text{-})^\ddagger$ gives a solution:

$$\begin{aligned} e^\ddagger &= \rho \cdot (\sigma \bullet e)^\dagger && \text{(def. of } (\text{-})^\ddagger) \\ &= \rho \cdot [\mathsf{id}, a] \cdot (\mathsf{id} + H(\sigma \bullet e)^\dagger) \cdot (\sigma \bullet e) && \text{(solution)} \\ &= [\rho, \rho \cdot a] \cdot (\mathsf{id} + H(\sigma \bullet e)^\dagger) \cdot (\sigma \bullet e) && \text{(coproduct)} \\ &= [\rho, b \cdot H\rho] \cdot (\mathsf{id} + H(\sigma \bullet e)^\dagger) \cdot (\sigma \bullet e) && (\rho \text{ is a homomorphism}) \\ &= [\mathsf{id}, b \cdot H\rho] \cdot (\rho + H(\sigma \bullet e)^\dagger) \cdot (\sigma + \mathsf{id}) \cdot e && \text{(coproduct, def. of } \bullet\text{)} \\ &= [\mathsf{id}, b \cdot H\rho] \cdot (\mathsf{id} + H(\sigma \bullet e)^\dagger) \cdot ((\rho \cdot \sigma) + \mathsf{id}) \cdot e && \text{(coproduct)} \\ &= [\mathsf{id}, b \cdot H\rho] \cdot (\mathsf{id} + H(\sigma \bullet e)^\dagger) \cdot e && \text{(section-retraction)} \\ &= [\mathsf{id}, b] \cdot (\mathsf{id} + H(\rho \cdot (\sigma \bullet e)^\dagger)) \cdot e && \text{(coproduct, functor)} \\ &= [\mathsf{id}, b] \cdot (\mathsf{id} + He^\ddagger) \cdot e && \text{(def. of } (\text{-})^\ddagger) \end{aligned}$$

(C) $(\text{-})^\ddagger$ is functorial: Let $e : X \to B + HX$, $f : Y \to B + HY$, and $h : X \to Y$ be a $(B + H(\text{-}))$-coalgebra homomorphism $\langle X, e\rangle \to \langle Y, f\rangle$. First, we notice that h is also a homomorphism between $(A + H(\text{-}))$-coalgebras $\langle X, \sigma \bullet e\rangle$ and $\langle Y, \sigma \bullet f\rangle$:

$$\begin{aligned} (\sigma \bullet f) \cdot h &= (\sigma + \mathsf{id}) \cdot f \cdot h && \text{(def. of } \bullet\text{)} \\ &= (\sigma + \mathsf{id}) \cdot (\mathsf{id} + Hh) \cdot e && (h \text{ homomorphism}) \\ &= (\mathsf{id} + Hh) \cdot (\sigma + \mathsf{id}) \cdot e && \text{(coproduct)} \\ &= (\mathsf{id} + Hh) \cdot (\sigma \bullet e) && \text{(def. of } \bullet\text{)} \end{aligned}$$

To show functoriality of $(\text{-})^\ddagger$:

$$\begin{aligned}
e^\ddagger &= \rho \cdot (\sigma \bullet e)^\dagger && \text{(def. of } (\text{-})^\ddagger) \\
&= \rho \cdot (\sigma \bullet f)^\dagger \cdot h && \text{(functoriality of } (\text{-})^\dagger) \\
&= f^\ddagger \cdot h && \text{(def. of } (\text{-})^\ddagger)
\end{aligned}$$

(D) $(\text{-})^\ddagger$ is compositional: Let $e : X \to Y + HX$ and $f : Y \to B + HY$. Then:

$$\begin{aligned}
(f^\ddagger \bullet e)^\ddagger &= \rho \cdot (\sigma \bullet (\rho \cdot (\sigma \bullet f)^\dagger) \bullet e)^\dagger && \text{(def. of } (\text{-})^\ddagger) \\
&= \rho \cdot ((\sigma \cdot \rho) \bullet (\sigma \bullet f)^\dagger \bullet e)^\dagger && \text{(props. of } \bullet) \\
&= \rho \cdot ((\sigma \bullet f)^\dagger \bullet e)^\dagger && \text{(A)} \\
&= \rho \cdot ((\sigma \bullet f) \blacksquare e)^\dagger \cdot \mathsf{inr} && \text{(compositionality of } (\text{-})^\dagger) \\
&= \rho \cdot (\sigma \bullet (f \blacksquare e))^\dagger \cdot \mathsf{inr} && \text{(props. of } \bullet \text{ and } \blacksquare) \\
&= (f \blacksquare e)^\ddagger \cdot \mathsf{inr} && \text{(def. of } (\text{-})^\ddagger)
\end{aligned}$$

(E) ρ preserves solutions:

$$\begin{aligned}
(\rho \bullet e)^\ddagger &= \rho \cdot (\sigma \bullet (\rho \bullet e))^\dagger && \text{(def. of } (\text{-})^\ddagger) \\
&= \rho \cdot ((\sigma \cdot \rho) \bullet e)^\dagger && \text{(properties of } \bullet) \\
&= \rho \cdot e^\dagger && \text{(A)}
\end{aligned}$$

Proof of Theorem 4

Let $\eta_A : A \to H^\infty A$ denote the canonical injection associated with the free object. Let $\rho : H^\infty A \to A$ be the unique homomorphism of Elgot algebras (that is, a solution-preserving morphism) such that $\mathsf{id} = \rho \cdot \eta_A$ obtained from the freeness of $\mathbf{F}A$. Since solution-preserving morphisms are homomorphisms of H-algebras, to see that ρ is an iteration-congruent retraction, it is left to check that it is indeed a congruence. For this, assume $\rho \bullet e = \rho \bullet f$ for some morphisms $e, f : X \to H^\infty A + HX$. Then:

$$\begin{aligned}
\rho \cdot e^\dagger &= (\rho \bullet e)^\ddagger && (\rho \text{ preserves solutions}) \\
&= (\rho \bullet f)^\ddagger && \text{(assumption)} \\
&= \rho \cdot f^\dagger && (\rho \text{ preserves solutions})
\end{aligned}$$

We also need to check that the $(\text{-})^\ddagger$ operator is indeed equal to the one obtained by the construction from Theorem 2:

$$\begin{aligned}
e^\ddagger &= ((\rho \cdot \eta_A) \bullet e)^\ddagger && \text{(section-retraction)} \\
&= (\rho \bullet (\eta_A \bullet e))^\ddagger && \text{(props. of } \bullet) \\
&= \rho \cdot (\eta_A \bullet e)^\dagger && (\rho \text{ preserves solutions})
\end{aligned}$$