Fuzzy logic

Radosław Warzocha

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1. Introduction

A *fuzzy concept* appearing in works of many philosophers, eg. Hegel, Nietzche, Marx and Engels, is a concept the value of which can vary according to context and conditions. We can think of those concepts as if they can be neither completely true not completely false. For example when can we say if someone is considered tall? For a person 1.2m or 2.1m tall the answer is obvious but what about someone who is 1.7m tall?

In 1920's Tarski and Łukasiewicz were studying multi-valued and infinite-valued logics, but the title of father of fuzzy logic is granted Lofti A. Zadeh who in 1965 presented the article *"Fuzzy sets"* where he described his fuzzy set theory – the foundation on which fuzzy logic was built.

2. Fuzzy sets

2.1. Definition

A fuzzy set $A = (\mathcal{U}, m_A)$ is a pair where $\mathcal{U}$ is a standard set, as we know them from set theory, and $m_A : \mathcal{U} \to [0; 1]$ is a membership function. A value $m_A(x)$ is called the *grade of membership* of x. We say that x is

- included in A when $m_A(x) > 0$
- excluded from A when $m_A(x) = 0$
- fully included in A when $m_A(x) = 1$

Sometimes, when considering fuzzy sets it is convenient to speak of

- kernel of A - $\{x \in \mathcal{U} \mid m_A(x) = 1\}$, and
- support of A - $\{x \in \mathcal{U} \mid m_A(x) > 0\}$.

We say that a fuzzy set A is empty when its support is $\emptyset$.

2.2. Set operations

Now we would like to define operations on fuzzy sets as well as a way of comparing them. We can define sum and product of sets in many ways. Some of them are presented in the following table:

Operators	**Sum** ($m_{A\cup B}(x)$)	**Product** ($m_{A\cap B}(x)$)
Zadeh	$\max(m_A(x), m_B(x))$	$\min(m_A(x), m_B(x))$
Product	$m_A(x) \cdot m_B(x)$	$m_A(x) + m_B(x) - m_A(x) \cdot m_B(x)$
Hamaher	$\dfrac{m_A(x) \cdot m_B(x)}{m_A(x) + m_B(x) - m_A(x) \cdot m_B(x)}$	$\dfrac{m_A(x) + m_B(x) - 2m_A(x) \cdot m_B(x)}{1 - m_A(x) \cdot m_B(x)}$
Einstein	$\dfrac{m_A(x) \cdot m_B(x)}{2 - (m_A(x) + m_B(x) - m_A(x) \cdot m_B(x))}$	$\dfrac{m_A(x) + m_B(x)}{1 + m_A(x) \cdot m_B(x)}$
Drastic	$\begin{cases} min(m_A(x), m_B(x)) & max(m_A(x), m_B(x)) = 1 \\ 0 & otherwise \end{cases}$	$\begin{cases} max(m_A(x), m_B(x)) & min(m_A(x), m_B(x)) = 0 \\ 1 & otherwise \end{cases}$
Bounded diff.	$\max(0, m_A(x) + m_B(x))$	$\min(1, m_A(x) + m_B(x))$

Table 1: Various definitions of operations on fuzzy sets

Choice of appropriate operator depends on purpose of the system (we can see practical example in section 4). We can also define set complement as a fuzzy set with membership function $m_{A'} = 1 - m_A$.

We say that $A \subseteq B$ if $m_A(x) \leq m_B(x)$ for every x in $\mathcal{U}$.

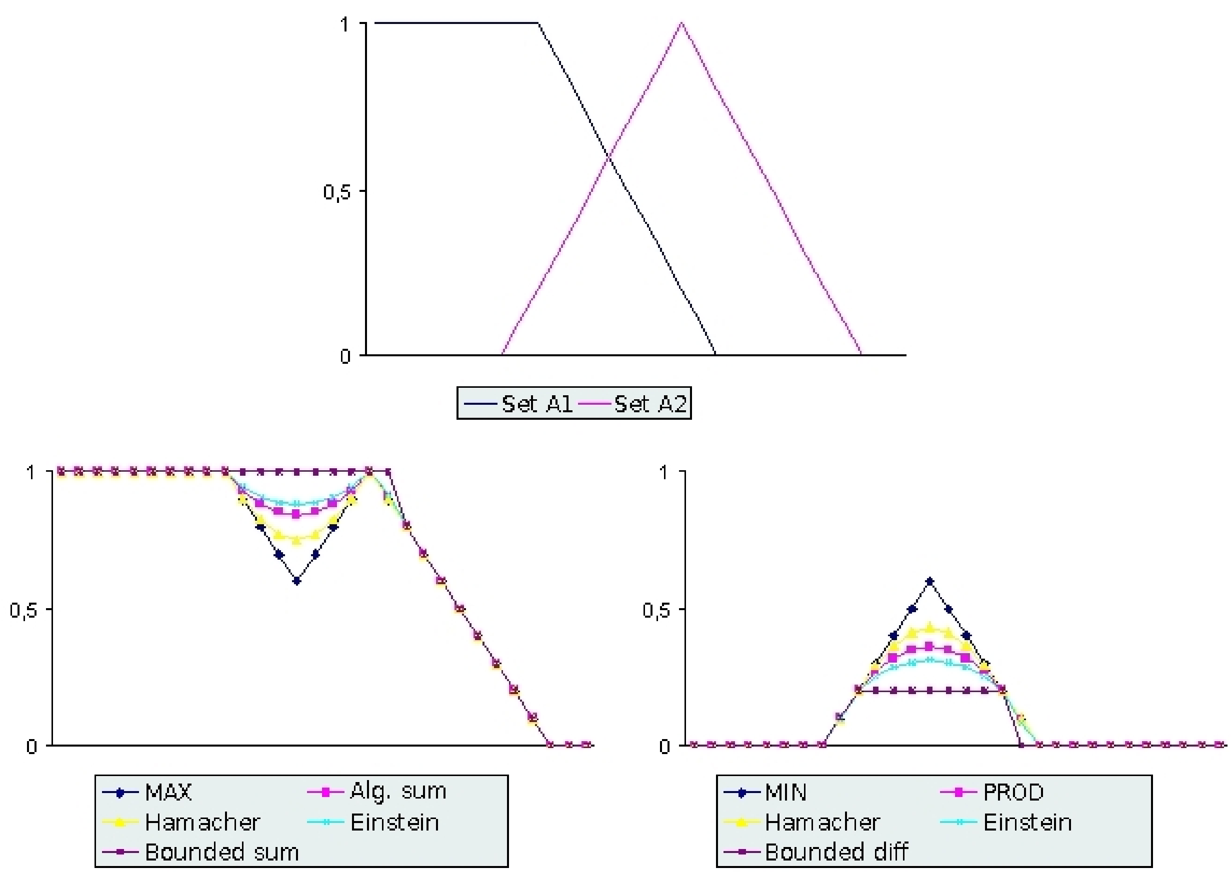

Figure 1: An illustration of various sum and product operators

2.3. Other definitions

Many classical concepts can be adapted to fit fuzzy set theory. For example convex combination of two vectors can be generalized to fuzzy sets in the following manner.

Definition
Let A, B and Λ be fuzzy sets. The *convex combination* of A, B and Λ is denoted by $(A, B; \Lambda)$ and is defined by its membership function

$$m_{(A,B;\Lambda)} = m_A m_\Lambda + m_B m_{\Lambda'}$$

Thanks to the definition we can see that convex combination of A, B and Λ satisfies the following property

$$A \cap B \subset (A, B; \Lambda) \subset A \cup B$$

We can also observe that, given any fuzzy set C satisfying $A \cap B \subset C \subset A \cup B$ one can always find Λ such that $C = (A, B; \Lambda)$.

We can generalize the concept of relation to embrace fuzziness as well. This extension will let us for example treat $x \ll y$ as fuzzy relation, which is a much more natural way than treating it as relation in the classic meaning.

Definition
A *n-ary fuzzy relation* R on sets $X_1, X_2, ..., X_n$ is a fuzzy set $(X_1 \times X_2 \times ... \times X_n, m_R)$ where m_R is a membership function of the form $m_R : X_1 \times X_2 \times ... \times X_n \to [0; 1]$.

Of course we should also define composition of binary relations $B \circ A$ as a fuzzy relation with the following membership function:

$$m_{B \circ A}(x, y) = \sup_{z \in \mathcal{U}} \min(m_A(x, z), m_B(z, y))$$

Finally, we'll see how we can generalize convexity on fuzzy sets.

Definition
We say that fuzzy set A is *convex* iff.

$$\forall \alpha \in [0; 1].\ \Gamma_\alpha = \{x \mid m_A(x) \geq \alpha\}$$

is convex. Alternatively iff.

$$\forall a_1, a_2 \in A. \forall \lambda \in [0; 1].\ m_A(\lambda a_1 + (1 - \lambda) a_2) \geq \min(m_A(a_1), m_A(a_2))$$

2.4. Fuzzy mathematics

Many other branches of fuzzy mathematics were developed on top of fuzzy set theory like fuzzy topology (Chang, 1968), group theory (Rosenfeld, 1971) or even graph theory (Kaufman, Rosenfeld, Yeh, 2000).

Also, the concept of a fuzzy number has been introduced:

Definition
A fuzzy number is a convex fuzzy set A with a segmentally continuous membership function which satisfies $\exists! x \in U. m_A(x) = 1$.

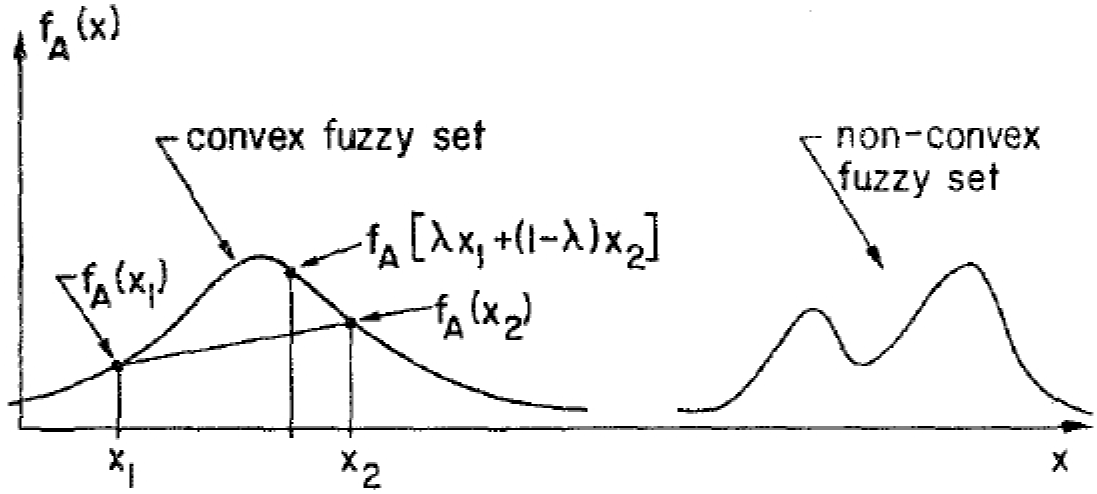

Figure 2: Convex and nonconvex fuzzy set

3. Formal systems

3.1. T-norms

<u>**Definition**</u>
A binary operator T which is

- commutative, i.e. $T(a,b) = T(b,a)$
- monotonic, i.e. if $a \leq c$ and $b \leq d$, then $T(a,b) \leq T(c,d)$
- associative, i.e. $T(a, T(b,c)) = T(T(a,b),c)$
- 1 is identity element, i.e. $T(a,1) = a$

is called a **triangular norm** (or a t-norm for short).
We can also say that t-norm is

- *strict* if it is strictly monotone
- *nilpotent* if it is continuous and $\forall x \in (0;1) \exists n \in \mathbb{N}.x^n = 0$
- *archimedean* if it is continuous and $\forall x, y \in (0;1).\exists n \in \mathbb{N}.x^n \leq y$

As we will shortly see, a t-norm properties make it a very good candidate to represent conjunction in our formal system. Classical conjunction is also commutative and associative. The property of monotonicity tells us that the overall 'truth value' becomes greater with the increase of the conjuncts' 'truth values'. The last property sets 1 as 'completely true' (and of course 0 as 'completely false').

But for our system to do its job conjunction is not enough. Let us define yet another binary operator called **residuum**.

DEFINITION

Let $*$ be a left-continuous t-norm. We define a **residuum of** $*$, denoted $\Rightarrow$, as

$$x \Rightarrow y := sup\{z | z * x \leq y\}$$

Let us take a look at some t-norms and their residua. We can see that they correspond to product operators on fuzzy sets

Name	**t-norm**	**residuum** (value for $x < y$)
Gödel	$min(x, y)$	y
Product	$x \cdot y$	y/x
Łukasiewicz	$max(0, x + y - 1)$	$1 - x + y$
Nilpotent	$\begin{cases} min(x, y) & x + y > 1 \\ 0 & otherwise \end{cases}$	$max(1 - x, y)$

Table 2: t-norms and residua defined by them

We can see that t-norm $*$ and it's residuum $\Rightarrow$ satisfy following conditions:

- $(x \Rightarrow y) = 1$ iff. $x \leq y$
- $(1 \Rightarrow y) = y$
- $min(x, y) \geq x * (x \Rightarrow y)$ (we can replace $\geq$ with $=$ when $*$ is continuous)
- $max(x, y) = min((x \Rightarrow y) \Rightarrow y, (y \Rightarrow x) \Rightarrow x)$

Now we want to see how to use those concepts in defining semantics of a formal logical system.

3.2. Monoidal t-norm logic (MTL)

Monoidal t-norm logic (or MTL for short) is a system described by Esera and Godo in 2001. It has the following syntactic constructs:

- $\rightarrow$ - implication
- & - strong conjunction
- $\wedge$ - weak conjunction
- $\bot$ - bottom
- $\neg$ - negation, $\leftrightarrow$ - equivalence, $\vee$ - (weak) disjunction and $\top$ - top

Now to define semantics we are going to define MTL algebra and its interpretation.

A **MTL algebra** is an algebraic structure $(L, \vee, \wedge, *, \Rightarrow, 0, 1)$ which satisfies the following properties:

- $(L, \vee, \wedge, 0, 1)$ is a bounded lattice (0 and 1 are minimal and maximal elements)

- $(L, *, 1)$ is a commutative monoid
- $z * x \leq y$ iff $z \leq (x \Rightarrow y)$
- $\forall x, y \in L.(x \Rightarrow y) \vee (y \Rightarrow x) = 1$

Furthermore, if $L = [0; 1], \vee = min, \wedge = max$, than we call this structure a **standard** MTL-algebra.

The interpretation of this structure is quite straightforward:

- t-norm $*$ corresponds to strong conjunction
- residuum $\Rightarrow$ corresponds to implication
- weak conjunction and disjunction correspond to lattice operators
- we define:

  $x \Leftrightarrow y \equiv (x \Rightarrow y) * (y \Rightarrow x)$

  $\neg x \equiv x \Rightarrow 0$

Formula A is said to be valid in a MTL-algebra if $e(A) = 1$ for every evaluation e. If A is valid in any MTL-algebra, then it is called a tautology.

Finally let us present a proof system we can use. Hilbert-style deduction system for fuzzy logic has only one rule (*modus ponens*) and 9 axioms:

Modus ponens: $(A\&(A \to B)) \to B$

1. $(A \to B) \to ((B \to C) \to (A \to C))$
2. $A\&B \to A$
3. $A\&B \to B\&A$

4a. $A \wedge B \to A$

4b. $A \wedge B \to B \wedge A$

4c. $A\&(A \to B) \to A \wedge B$

5a. $(A \to (B \to C)) \to (A\&B \to C)$

5b. $(A\&B \to C) \to (A \to (B \to C))$

6. $((A \to B) \to C) \to (((B \to A) \to C) \to C)$
7. $\bot \to A$

Now let us show proof of a simple fact: $A \to A$, in order to get better understanding of the presented system.

1. $((A \to A)\&A \to A \wedge A) \to ((A \wedge A \to A) \to ((A \to A)\&A \to A))$	(Axiom 1)
2. $(A \to A)\&A \to A \wedge A$	(Axiom 4c)
3. $(A \wedge A \to A) \to ((A \to A)\&A \to A)$	(modus ponens for 1 and 2)
4. $A \wedge A \to A$	(Axiom 4a)
5. $(A \to A)\&A \to A$	(modus ponens for 3 and 4)
6. $((A \to A)\&A \to A) \to ((A \to A) \to (A \to A))$	(Axiom 5b)
7. $(A \to A) \to (A \to A)$	(modus ponens for 5 and 6)
8. $((A \to A) \to (A \to A)) \to (((A \to A) \to (A \to A)) \to (A \to A))$	(Axiom 6)
9. $((A \to A) \to (A \to A)) \to (A \to A)$	(modus ponens for 7 and 8)
10. $(A \to A)$	(modus ponens for 7 and 9)

Some other proof systems, like a hypersequent system for fuzzy logic, are presented in [3].

3.3. Extensions and other logics

As we may observe, negation defined in this way does not obey the law of double negation and because of that sometimes we want to introduce involutive negation $\sim$, to our system. We can also extend it with *strong disjunction* defined as $A \oplus B \equiv \sim (\sim A \& \sim B)$.

There are other ways of defining fuzzy logic of which the most important is Basic propositional fuzzy logic (BL) proposed by Hájek in 1998. The difference between Bl and MTL is that in BL we need our t-norm to be continuous (not only left-continuous like in MTL). We can also talk about Gödel fuzzy logic, Łukasiewicz fuzzy logic etc. depending on the t-norm we choose. We can also define predicate fuzzy logic or even intuitionistic fuzzy logic (IF).

4. Applications

Fuzzy logic is mostly used in fuzzy controllers and AI. However, hardware controllers should be easily programmable. For this reason fuzzy logic is usually programmed with IF-THEN clauses or an equivalent concept (like fuzzy associative matrices). An example program has the following form:

```
IF temperature IS cold OR humidity IS dry
   THEN low fan speed
IF temperature IS normal
   THEN medium fan speed
IF temperature IS hot AND humidity IS wet
   THEN high fan speed
```

There is no 'else' clause, and all of the options are evaluated. Many rules can be applied at once. To generate a final response the program needs to use so called *defuzzyfication method* – function which, when given fuzzy sets and membership degrees corresponding to them, produces quantifiable result. Let us show an example of the defuzzyfication method, called Center of Gravity (or COG for short).

What COG first does is it translates input sets (in our case *temperature-cold, humidity-dry* and so on) into output sets (again, in our case *fan speed-low* and so on). For example let us consider first clause in program shown above:

```
IF temperature IS cold OR humidity IS dry
   THEN low fan speed
```

Let x, y be current values of temperature and humidity respectively, also let m_{tl}, m_{hd} and m_{fl} be membership functions of sets corresponding to low temperature, dry air and low fan speed. In GOC we take values $m_{tl}(x)$ and $m_{hd}(y)$ and then create new fuzzy set with membership function

$$m'_{fl}(z) = min(m_{fl}(z), max(m_{tl}(x), m_{hd}(y)))$$

Of course other function then *max* can be applied depending on the choice of appropriate fuzzy operators. Also `AND` condition in the clause would make use of the product operator instead of sum operator.
We can see illustration of this process in figure 3.

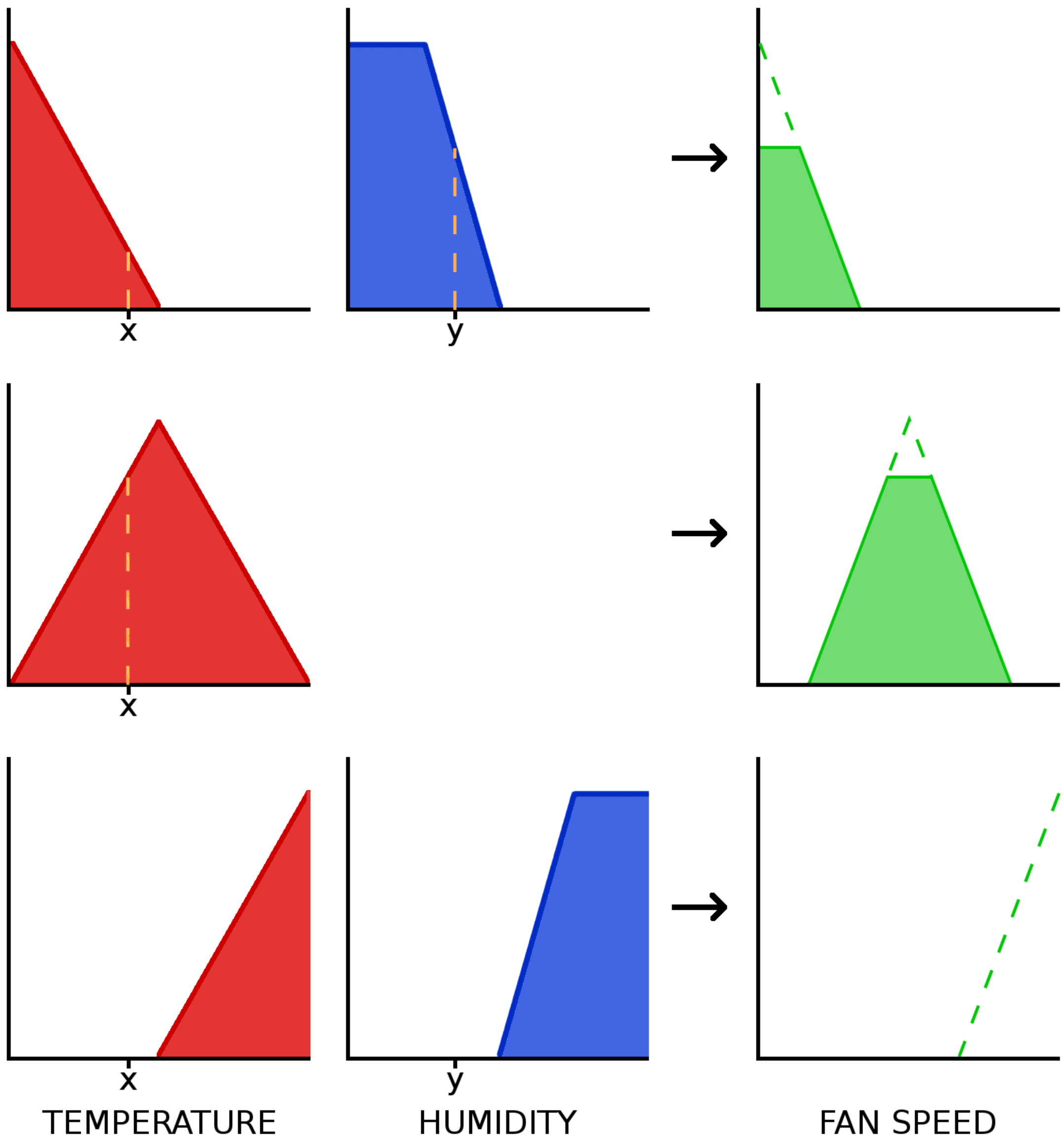

Figure 3: Translation of the input sets into the output sets.

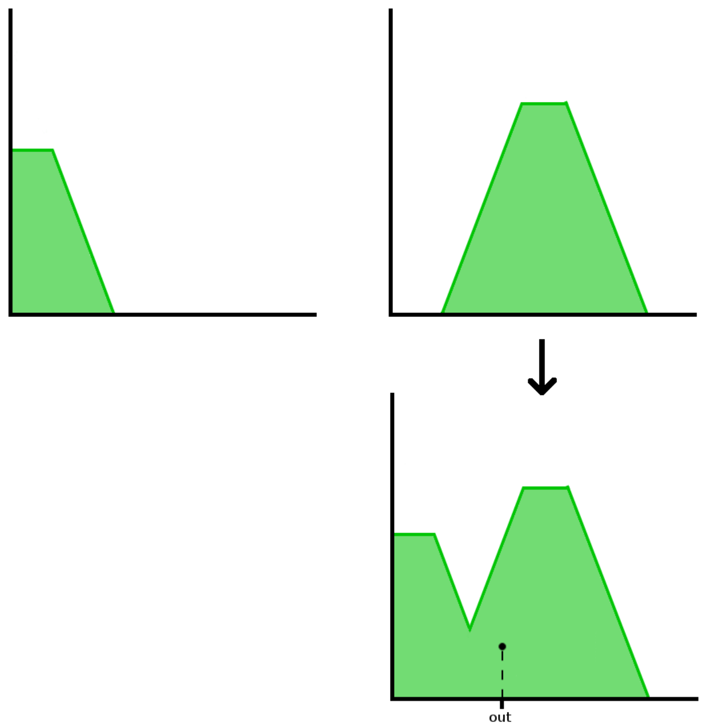

Figure 4: Accumulating output sets and calculating centroid.

Next we accumulate those new sets (there is one for each clause) by taking their sum and obtain yet another fuzzy set, which membership function we will denote by m_{acc}.
Finally we compute centroid of the area bounded by the plot of m_{acc}. One can find abscissa of the centroid using following formula:

$$C_x = \frac{\int x m_{acc}(x) dx}{\int m_{acc}(x) dx}$$

Note that in most cases the integrals appearing in formula above, thanks to the common 'shape' of fuzzy sets, should be quite easy to compute. Obtained value C_x is final response of the COG method.

There are many other commonly used methods of defuzzification, including:

- COA (center of area)
- FM (fuzzy mean)
- FOM (first of maximum)

Choice of appropriate method should be based on purpose of the system and past experience.

5. Bibliography

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[3] George Metcalfe, *"Proof theory for propositional fuzzy logic"*, http://sitemason.vanderbilt.edu/files/fAXPwI/thesis.ps