Results on the Guarded Fragment with Equivalence or Transitive Relations

Emanuel Kieroński

Institute of Computer Science, University of Wrocław

Abstract. We study the problem of the satisfiability of guarded formulas in models in which some distinguished binary symbols are interpreted as equivalence relations or as transitive relations. We sharpen the undecidability result for the two-variable guarded fragment with transitive relations by reducing the number of transitive relations to two. We prove that the satisfiability problem for the two-variable guarded fragment with two equivalence relations is 2EXPTIME-complete. We consider the guarded fragment with equivalence relations in guards and show that this variant is easily reducible to the variant with transitive relations in guards. However, in the case of two variables, the version with equivalence relations is easier: NEXPTIME-complete. Finally we show that the decidability results for the guarded fragment with either equivalence relations or transitive relations in guards cannot be generalized to the loosely guarded fragment.

1 Introduction

The *guarded fragment* GF is a fragment of first-order logic[1] FO in which the usage of quantifiers is restricted. GF is defined inductively: all atomic formulas belong to GF; GF is closed under Boolean connectives; if $\varphi(\mathbf{x},\mathbf{y}) \in$ GF then $\forall\mathbf{x}(\alpha(\mathbf{x},\mathbf{y}) \rightarrow \varphi(\mathbf{x},\mathbf{y}))$ and $\exists\mathbf{x}(\alpha(\mathbf{x},\mathbf{y}) \wedge \varphi(\mathbf{x},\mathbf{y}))$ belong to GF, where $\alpha(\mathbf{x},\mathbf{y})$ is an atomic formula containing all the free variables of φ. In particular, α may be of the form $x = x$. Atoms α are called *guards*.

The guarded fragment was introduced by Andréka, van Benthem and Németi [1] as a generalization of propositional modal logic. Since then various variants and extensions of GF have been investigated. It appeared that GF retains a lot of good properties of modal logics. In particular its satisfiability problem is decidable [1, 4], it has the finite model property and (a kind of) tree model property [4], remains decidable when augmented with fixed-point operators [8].

Since the transitivity of a binary relation is not expressible in GF, to obtain counterparts of some modal logics with axioms of transitivity or equivalence, like K4 or S5, it is natural to study the satisfiability of the guarded fragment in restricted classes of models, in which some binary relations are interpreted as transitive or, respectively, equivalence relations. In this paper we follow the convention, that when considering the guarded fragment, this requirement is expressed syntactically: we "extend" the guarded fragment by the possibility of stating that some binary relations have to be transitive

[1] In this paper we consider first-order logic without constants and function symbols.

(equivalence) relations. Now, a formula consists of its purely guarded part and a list $trans[T_1, \ldots, T_l]$ ($equiv[T_1, \ldots, T_l]$) of transitive (equivalence) relations.[2]

Unfortunately, it appeared that the guarded fragment with transitive relations GF+ $trans$ is undecidable. The first proof of this fact was given in [4], where undecidability of GF^3+$trans[T_1, T_2]$, the guarded fragment with three variables and two transitive relations, was established. Since modal logic can be embedded into the two-variable guarded fragment GF^2, it was very interesting what happens on the level of two variables. This question was partially answered by Ganzinger, Meyer and Veanes [3] who showed undecidability of GF^2+$trans[T_1, \ldots, T_4]$ and GF^2+$trans[T_1, \ldots, T_5]$ without equality. In this paper we sharpen this result and show undecidability of GF^2+ $trans[T_1, T_2]$ without equality and with T_1, T_2 being the only non-unary symbols. Moreover, we believe, that in spite of restricting the language, our proof is simpler than the proof in [3]. The result is optimal with respect to the number of transitive relations, since, as we argue, GF^2+$trans[T]$ is decidable, and in fact 2EXPTIME-complete.

The undecidability of GF^2+$trans[T_1, T_2]$ becomes even more interesting if compared to the results on the two-variable guarded fragment with equivalence relations. In [11] we proved undecidability of GF^2+$equiv[E_1, E_2, E_3]$ without equality, but it appeared that even the whole FO^2+$equiv[E_1, E_2]$ is decidable. Because of the high level of technical complication we were not able to give precise complexity bounds for the last variant. On the level of the guarded fragment the situation is easier and in this paper we show that GF^2+$equiv[E_1, E_2]$ is 2EXPTIME-complete. This reveals, that reasoning about two equivalence relations is more difficult than about one, since even FO^2+$equiv[E]$ is in NEXPTIME [11].

The undecidability of the guarded fragment with several transitive or equivalence relations seems to be quite surprising in the context of connections to modal logics: modal logics K4 and S5 are decidable even in their multi-modal variants involving more than one transitive or equivalence relation. To explain this, we should note that in the natural translation of modal logic into the guarded fragment all these special relations appear only in guards. And indeed, the guarded fragment with *transitive guards* GF+TG, the version of GF+$trans$ in which only the relations appearing only in guards can be required to be transitive, is decidable in 2EXPTIME [15]. See [3, 15, 9, 10, 16] for more discussions and results on this variant.

Here we consider the guarded fragment with equivalence relations in guards GF+EG and show that it is easily reducible to GF+TG. For two variables, however, GF^2+EG is easier than GF^2+TG. We show that it is NEXPTIME-complete, while the latter was shown to be 2EXPTIME-hard [10].

In the Table 1 we summarize the results on the satisfiability of the two-variable guarded fragment with transitive or equivalence relations.

In the final part of this paper we look at a more liberal extension of the guarded fragment, the so-called *loosely* guarded fragment, LGF. In LGF [2], the notion of guard is relaxed. A guard $\alpha(\mathbf{x}, \mathbf{y})$ may be a conjunction of atoms provided that each pair of variables from $\mathbf{x} \cup \mathbf{y}$ appears together in an atom. In [3] it was observed that LGF^3 with one transitive relation is undecidable. The proof went by a reduction from the inter-

[2] Note that the set of formulas obtained in this way is not really what we usually call "logic"; for example it is not closed under negation.

Table 1. Satisfiability of GF^2 with transitive or equivalence relations

type of special relations	only in guards	# of equivalence/transitive relations		
		1	2	3 or more
equivalence	NEXPTIME	NEXPTIME	2EXPTIME	undecidable
transitive	2EXPTIME	2EXPTIME	undecidable	undecidable

section emptiness problem for context-free languages. Here we give a straightforward, very easy proof of the slightly stronger result: undecidability of LGF^3 without equality with one transitive relation which is used only in guards. Similar result can be also obtained for LGF^3 with one equivalence relation in guards. Moreover, in the case of one transitive relation, we can make an effort to use the transitive symbol as the only non-unary symbol in our proof. These results show that the distinction between *relation in guards only* and *relations everywhere*, which is important for transitive and equivalence relations in GF, disappears in LGF.

Besides obvious connections to modal logics, the study of the satisfiability of the guarded fragment in restricted classes of models can be also motivated by similar study for FO^2. See [6, 7, 13, 11].

2 Plan of the Paper

In Section 3 we adapt the normal form theorems for the guarded fragment to our purposes. Section 4 contains results on GF^2+$trans$. In Section 5 we study GF+EG, in particular GF^2+EG, and prove that the GF^2+$equiv[E_1, E_2]$ is 2EXPTIME-complete. In Section 6 we show undecidability of LGF^3+TG$[T]$ and LGF^3+EG$[T]$.

A note for the reader. We assume that the reader knows basic concepts from model theory and theoretical computer science, in particular notions of (atomic) 1-types and 2-types and results on alternating Turing machines.

The paper is closely related to [15] and [11]. In a few places we refer not only to results from these papers but also to techniques used in proofs. In such parts we do not present whole constructions in detail, but rather sketch main ideas.

3 Normal Forms

We want to adapt the normal form theorems for the guarded fragment to make them more convenient to our purposes. Such theorems were proved by Grädel in [4] and Szwast and Tendera in [15]. Let us review the results from [15].

Definition 1. *We say that a* GF^2 *sentence is in normal form if it is a conjunction of sentences of the following form:*

- $\exists x\ (\alpha(x) \wedge \chi(x))$,
- $\forall x\ (\alpha(x) \rightarrow \exists y\ ((\beta(x,y) \wedge \chi(x,y)))$,
- $\forall x \forall y\ (\beta(x,y) \rightarrow \chi(x,y))$,

where $\alpha(x)$, $\beta(x,y)$ are atomic formulas and χ is quantifier-free.

It can be shown:

Theorem 1 (Szwast, Tendera). *With every GF^2 sentence φ of the length n one can effectively associate a set Δ of GF^2 sentences in normal form (over the extended vocabulary), $\Delta = \{\delta_1, \ldots, \delta_d\}$ such that*
(1) *φ is satisfiable if and only if $\bigvee_{i\leq d} \delta_i$ is satisfiable,*
(2) *$d \leq O(2^n)$ and for every $i \leq d$, $|\delta_i| = O(n \log n)$,*
(3) *Δ can be computed deterministically in exponential time and every sentence δ_i can be computed in time polynomial with respect to n,*
(4) *$\bigvee_{i\leq d} \delta_i \models \varphi$ and every model of φ can be expanded to a model of $\bigvee_{i\leq d} \delta_i$.*

The additional property of normal form of Szwast and Tendera is that if a binary symbol T appears in φ only in guards then it is also the case in Δ. It is important when working with either the guarded fragment with transitive guards GF+TG or the guarded fragment with equivalence relations in guards GF+EG. We use this property in Section 5.1.

Let us introduce the new definition:

Definition 2. *A GF^2+$equiv[E_1, E_2]$ sentence φ is in $[E_1, E_2]$-*guarded normal form *if it is a conjunction of sentences of the following form:*

- $\exists x\ (\alpha(x) \wedge \chi(x))$,
- $\forall x\ (\alpha(x) \rightarrow \exists y\ (E_1xy \wedge E_2xy \wedge \chi(x,y)))$ ($\forall\exists^{++}$ *form*),
- $\forall x\ (\alpha(x) \rightarrow \exists y\ (E_1xy \wedge \neg E_2xy \wedge \chi(x,y)))$ ($\forall\exists^{+-}$ *form*),
- $\forall x\ (\alpha(x) \rightarrow \exists y\ (E_2xy \wedge \neg E_1xy \wedge \chi(x,y)))$ ($\forall\exists^{-+}$ *form*),
- $\forall x\ (\alpha(x) \rightarrow \exists y\ (\beta(x,y) \wedge \neg E_1xy \wedge \neg E_2xy \wedge \chi(x,y)))$ ($\forall\exists^{--}$ *form*),
- $\forall x \forall y\ (\beta(x,y) \rightarrow \chi(x,y))$ ($\forall\forall$ *form*),

where $\alpha(x)$, $\beta(x,y)$ are atomic formulas, and χ is quantifier-free.

The polynomial transformation from normal form from Definition 1 to normal form from Definition 2 is standard and we skip it here. The existence of such a transformation allows us to substitute the words *normal form* in Theorem 1 with $[E_1, E_2]$-*guarded normal form*.

4 Guarded Fragment with Transitive Relations

4.1 A Note on the Case of One Transitive Relation

Essentially, the decidability of GF^2+$trans[T]$ can be shown by applying the construction of Szwast and Tendera for GF^2+TG [15]. The only problem is that sometimes a conjunct of the form $\forall x\ (\alpha(x) \rightarrow \exists y\ (\beta(x,y) \wedge \chi(x,y)))$ without T in the guard β may say in χ that x and y have to be connected by T. To avoid this problem we first transform a formula into $[T]$-guarded normal form, which is similar to $[E_1, E_2]$-guarded normal form, with symbol T treated in a special way. The construction from [15] gives now the 2EXPTIME upper bound. The lower bound follows from [10]. We have:

Theorem 2. *The satisfiability problem for GF^2+$trans[T]$ is 2EXPTIME-complete.*

4.2 Two Transitive Relations

In this subsection and in Section 6 we use the technique from [6, 13]. It was originally designed for extensions of FO^2 but adaptations for extensions of GF^2 without equality are straightforward. Due to the space limit we give only the statement of the lemma we require, without a proof.

We say that an infinite structure $\mathfrak{G} = (G, H, V)$ is *grid-like* if the standard grid $\mathfrak{G}_{\mathbb{N}}$ is homomorphically embeddable into $\mathfrak{G}$; a finite $\mathfrak{G}$ is grid-like if some $\mathfrak{G}_m$ is homomorphically embeddable into $\mathfrak{G}$, where $\mathfrak{G}_m$ is the standard grid on $m \times m$ torus.

Lemma 1. *Let $\mathcal{L}$ be an extension of GF^2 without equality. Let N be a distinguished unary symbol of the vocabulary. If there exists an $\mathcal{L}$-formula φ such that:*
(1) *$\mathfrak{G}_{\mathbb{N}}$ can be expanded to a model of φ,*
(2) *for every $k \in \mathbb{N}$ there exists $m > k$ such that $\mathfrak{G}_m$ can be expanded to a model of φ,*
(3) *if $\mathfrak{A} \models \varphi$ is infinite then $\mathfrak{A}{\restriction}N$ is grid-like,*
(4) *if $\mathfrak{A} \models \varphi$ is finite then $\mathfrak{A}{\restriction}N$ is grid-like,*
then $\mathcal{L}$ forms a conservative reduction class. If at least conditions (1) and (3) hold then the satisfiability problem for $\mathcal{L}$ is undecidable.

We improve the undecidability result from [3] by showing that the two-variable guarded fragment with two transitive relations is undecidable. Actually we prove the stronger result, implying also the undecidability of the finite satisfiability problem:

Theorem 3. *GF^2+$trans[T_1, T_2]$ without equality is a conservative reduction class.*

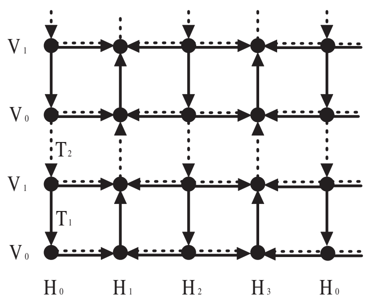

Fig. 1. Grid structure for GF^2+$trans[T_1, T_2]$

Proof. Let us expand the standard grid to the structure $\bar{\mathfrak{G}}_{\mathbb{N}}$ illustrated in the Fig. 1. Additionally, for the unary symbol N, Na is true for every element a of the structure. We capture some properties of $\bar{\mathfrak{G}}_{\mathbb{N}}$ by the sentence φ, then observe that every $\mathfrak{G}_{4m}$ can also be expanded to a model of φ and prove that the restriction to N of every model of φ is grid-like.

The sentence φ is the conjunction of the following formulas:

(A) The initial formulas:

$$\exists x\ (H_0x \wedge V_0x \wedge Nx), \tag{1}$$

$$\forall x\ (Nx \to \exists y\ (Hxy \wedge Ny)), \tag{2}$$

$$\forall x\ (Nx \to \exists y\ (Vxy \wedge Ny)). \tag{3}$$

(B) A formula axiomatising H, which has the following shape:

$$\forall xy\ (Hxy \to (\varphi_1 \vee \varphi_2 \vee \ldots \vee \varphi_8)), \tag{4}$$

where each φ_i describes one of the eight possible cases of values of unary predicates on H-connected vertices and transitive connections between them. For example:

$$\varphi_1 \equiv T_1xy \wedge T_2xy \wedge H_0x \wedge V_0x \wedge H_1y \wedge V_0y \wedge Nx \wedge Ny, \tag{5}$$

$$\varphi_2 \equiv T_1yx \wedge T_2yx \wedge H_1x \wedge V_0x \wedge H_2y \wedge V_0y \wedge Nx \wedge Ny. \tag{6}$$

(C) A formula axiomatising V, which is built similarly to the one for H.

(D) A group of formulas stating that some horizontally adjacent elements that are connected by T are also linked by H. Example formulas from this group are:

$$\forall xy\ (T_1xy \to ((H_0x \wedge V_1x \wedge H_1y \wedge V_1y) \to Hxy)), \tag{7}$$

$$\forall xy\ (T_2xy \to ((H_2x \wedge V_0x \wedge H_1y \wedge V_0y) \to Hyx)). \tag{8}$$

It is not hard to see that every $\mathfrak{G}_{4m}$ can be expanded to a model of φ. It is enough to take a natural quotient.

We sketch the argument for grid-likeness of $\mathfrak{A}' = \mathfrak{A}\restriction N$, for $\mathfrak{A} \models \varphi$. It is enough to show that H is complete over V in $\mathfrak{A}'$, i.e. $\mathfrak{A}' \models \forall xyx'y'((Hxy \wedge Vxx' \wedge Vyy') \to Hx'y')$ (cf. [13]). Assume that for a, a', b, b':

$$\mathfrak{A}' \models Hab \wedge Vaa' \wedge Vbb'.$$

We show that then $\mathfrak{A}' \models Ha'b'$. One should consider several cases distinguished by values of the H_i and V_i on a. Let us go through one of them, for instance $\mathfrak{A}' \models H_1a \wedge V_1a$. By (B) we have:

$$\mathfrak{A}' \models H_2b \wedge V_1b \wedge T_2ba.$$

Similarly (C) implies:

$$\mathfrak{A}' \models H_1a' \wedge V_0a' \wedge T_2aa'$$

and

$$\mathfrak{A}' \models H_2b' \wedge V_0b' \wedge T_2b'b.$$

From transitivity of T_2 it follows that $\mathfrak{A}' \models T_2b'a'$. Now an appropriate formula of the form (D) guarantees $\mathfrak{A}' \models Ha'b'$ which finishes the proof for this case. The remaining seven cases can be treated in the similar way. $\square$

Remark. The binary relation symbols H and V do not play the crucial role in the above proof. They can be simulated by T_1, T_2 and values of unary predicates H_i and V_i. Thus we obtain undecidability of GF^2+$trans[T_1, T_2]$ without equality, even if T_1 and T_2 are the only non-unary symbols allowed.

5 Guarded Fragment with Equivalence Relations

5.1 Equivalence Relations in Guards

In [15] Szwast and Tendera study the guarded fragment with transitive guards GF+TG, in which merely relations appearing only in guards may be required to be transitive. It is also natural to consider the guarded fragment with equivalence relations in guards GF+EG. We observe that there is a simple reduction from GF+EG to GF+TG.[3]

Lemma 2. *There is a polynomial time reduction which transforms a GF+EG sentence φ into a GF+TG sentence φ' such that φ is satisfiable if and only if φ' is satisfiable.*

Proof. Let $\varphi = \varphi' + equiv[E_1, \ldots, E_k]$ be a GF+EG sentence. We construct a GF+TG sentence ψ, in an extended vocabulary, which is satisfiable if and only if φ is satisfiable. For every $1 \leq i \leq k$ we introduce a new unary predicate P_i and put the formulas:

$$\forall xy\ (E_i xy \rightarrow ((P_i x \wedge P_i y) \rightarrow x = y)), \tag{9}$$

$$\forall x\ (x = x \rightarrow \exists y\ (E_i xy \wedge P_i y)) \wedge \forall x\ (x = x \rightarrow \exists y\ (E_i yx \wedge P_i y)). \tag{10}$$

We define ψ as the conjunction of φ', formulas (9)-(10) for $1 \leq i \leq k$, and the transitivity requirement: $trans[E_1, \ldots, E_k]$.

Now, every model of φ can be expanded to a model of ψ by choosing for every i exactly one element in each E_i-class, and marking it with P_i. On the other hand in every model of ψ each E_i has to be symmetric and reflexive and thus, because of transitivity of E_i, it has to be an equivalence relation. □

Since the satisfiability of GF+TG was shown in [15] to be in 2EXPTIME, and the satisfiability of the pure guarded fragment GF is 2EXPTIME-hard we have:

Corollary 1. *The satisfiability problem for GF+EG is 2EXPTIME-complete.*

Let us turn our attention to the case of the two-variable fragment and observe that GF^2+EG is easier than GF^2+TG which is complete for 2EXPTIME [10, 15].

Theorem 4. *The satisfiability problem for GF^2+EG is NEXPTIME-complete.*

Proof. **The Upper Bound.** The proof of the upper bound is similar to the proof of the upper bound for GF^2+TG given by Szwast and Tendera [15]. In their proof it is shown how to construct a *ramified* model for a satisfiable formula. The most important properties of a ramified model are:

- it is *tree-controlled*,
- each pair of elements is connected by at most one transitive relation (possibly in both directions),

[3] Notice that the straightforward idea of expressing symmetry of a transitive relation E: $\forall xy\ (Exy \rightarrow Eyx)$ leads to the formula in which E appears outside guards.

– the size of every T_i-clique,[4] for a transitive T_i, is at most exponential in the size of the input formula.

The similar properties may be obtained in our case. We sketch the construction. Let φ be a GF^2+EG sentence in normal form from Definition 1, with equivalence relations $E_1, \ldots E_k$. Let $\mathfrak{A} \models \varphi$. We build a new model $\mathfrak{B} \models \varphi$ which consists of (possibly) infinite number of layers. The Layer 0 consists of elements satisfying conjuncts of the form $\exists x(\alpha(x) \wedge \chi(x))$. The Layer 1 contains E_i-classes for $1 \leq i \leq k$ and free witnesses (witnesses for formulas without equivalence symbols in guards) for elements from the Layer 0. The E_i-classes of an element b may be constructed in such a way that for $i \neq j$, E_i-class of b and E_j-class of b have exactly one common element: b, and each E_i-class has size at most exponential in $|\varphi|$. The construction of these classes is the same as the construction of T_i-cliques of an element in the proof of Szwast and Tendera for GF^2+TG. We say that an E_i-class of b is its *petal*. All the free witnesses of b we put outside the petals of b. They are called *leaves*. Petals and leaves of b form the *flower* of b.

We proceed recursively. For an element b from the Layer i we build its flower and put its petals (with the exception of at most one which was built in the previous step) and leaves into the Layer (i+1). We define all non-specified 2-types in $\mathfrak{B}$ as negative types consistent with 1-types, where by a negative 2-type we mean a type without positive occurrences of binary predicates.

This way we obtain a ramified model $\mathfrak{B} \models \varphi$. For every 1-type t_i which is realized in $\mathfrak{B}$ we choose an element b_i of type t_i. We argue that a list L of such elements and its flowers is sufficient for construction of another ramified model $\mathfrak{B}' \models \varphi$. Observe that the information we require is of size exponential in $|\varphi|$. To construct $\mathfrak{B}'$ we put for every t_i in L a new copy of the flower of b_i into the Layer 0. Now, we proceed recursively again. Let b be an element in the Layer i whose flower is not defined. There may be at most one petal built for b, say E_r-petal. In our list we find b' of the same type as b and we put a new copy of every petal of b', except its E_r-petal, to the Layer (i+1). Similarly, we add all the required leaves for b.

The procedure deciding the satisfiability problem for GF^2+EG may be the following:

input: a GF^2+EG sentence φ
begin procedure
- compute the set Δ from Theorem 1,
- *guess* a sentence δ_i in normal form from Definition 1 from Δ,
- *guess* a list of 1-types realized in a ramified model of δ_i, and shapes of its flowers,
- check if a ramified model may be constructed basis of this list.

end procedure

Clearly, it works in nondeterministic exponential time.

The Lower Bound. The lower bound can be easily obtained without equality, even in the presence of only one equivalence relation E. Using two sets of unary predicates

[4] a T_i-clique is a maximal set X of elements such that for every $a, b \in X$ both $T_i(a, b)$ and $T_i(b, a)$ are true.

$H_1, \ldots H_n$ and $V_1, \ldots, V_n$ we can write a formula such that its model has to contain an E-class which can be viewed as a square of size $2^n \times 2^n$ such that for every element, H_i and V_i code in binary its horizontal and vertical position. Having such a class it is easy to simulate a nondeterministic Turing machine working in time exponential in the size of its input. $\square$

5.2 GF^2 with Two Equivalence Relations

In [11] we considered the whole FO^2 with equivalence relations. We proved that FO^2+ $equiv[E]$ has exponential model property. From this result and from the proof of the lower bound in Theorem 4 it follows that GF^2+$equiv[E]$ is NEXPTIME-complete. In [11] we proved also that the satisfiability problem for whole two-variable first order logic with two equivalence relations is decidable in 3NEXPTIME. Because of the high level of technical complication of our construction we were not able to give exact complexity bounds. In the case of the two-variable guarded fragment the situation is easier. However, we still have to take into account infinite models. The following example of an infinity axiom comes from [11]. If E_1 and E_2 are interpreted as equivalence relations then the conjunction φ of the formulas (11)-(16) is satisfiable only in infinite models.

$$\exists x\ (Px \wedge Sx) \tag{11}$$

$$\forall x\ (Px \rightarrow \exists y\ (E_1xy \wedge x \neq y \wedge Qy)) \tag{12}$$

$$\forall x\ (Qx \rightarrow \exists y\ (E_2xy \wedge x \neq y \wedge Py)) \tag{13}$$

$$\forall xy\ (E_1xy \rightarrow ((Px \wedge Py) \rightarrow x = y)) \tag{14}$$

$$\forall xy\ (E_2xy \rightarrow ((Qx \wedge Qy) \rightarrow x = y)) \tag{15}$$

$$\forall x\ (Sx \rightarrow \neg\exists y\ (E_2xy \wedge x \neq y)) \tag{16}$$

An example model is illustrated in the Fig. 2.

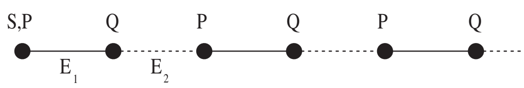

Fig. 2. A model of φ

Here we show:

Theorem 5. *The satisfiability problem for GF^2+$equiv[E_1, E_2]$ is 2EXPTIME-complete.*

Proof. **The Lower Bound.** The proof of the lower bound is similar in spirit to the proof of 2EXPTIME-lower bound for GF^2+TG from [10].

It suffices to show that every problem in AEXPSPACE can be reduced in polynomial time to the satisfiability problem for GF^2+$equiv[E_1, E_2]$. Let M be an alternating Turing machine working in space bounded by 2^{n^k}. Let w be an input for M. We construct a GF^2+$equiv[E_1, E_2]$ sentence φ which is satisfiable if and only if M accepts w. Without any loss of generality we can assume that in every configuration M has exactly two possible transitions and that it enters an accepting or rejecting state in exactly $2^{2^{n^k}}$-th step. Moreover, to simplify the proof, we assume that after this step M does not stop, but works infinitely. More precisely, the accepting and rejecting states are universal. In each of such states M has two identical transitions: it does not write any symbol on the tape and it does not change its state. In other words, after accepting or rejecting M stays infinitely in the same configuration. Such assumption about M allows us not to bother about the numbering of configurations.

Every configuration is represented by a set of 2^{n^k} elements, each of them corresponding to a single cell of the tape. To encode a position of an element in a configuration, i.e. a consecutive number of a tape cell it represents, we use the unary relation symbols $P_1, \ldots, P_{n^k}$. Formally, $P_i a$ is true if the i-th bit of the position of a is set to 1. We use the abbreviation $\overline{P}(a)$ to describe this position ($0 \leq \overline{P}(a) < 2^{n^k}$).

The following properties (for fixed l, $0 \leq l < 2^{n^k}$) can be easily expressed by quantifier-free formulas of the length polynomial in n:

$$\overline{P}(x) = l, \quad \overline{P}(x) \geq l, \quad \overline{P}(x) = \overline{P}(y), \quad \overline{P}(x) = \overline{P}(y) + 1.$$

For example, the last property can be expressed as follows:

$$\bigvee_{0 \leq i < n^k} (P_i x \wedge \neg P_i y \wedge \bigwedge_{j > i} (\neg P_j x \wedge P_j y) \wedge \bigwedge_{j < i} (P_j x \leftrightarrow P_j y)).$$

We connect each pair of elements a, b belonging to the representation of a same configuration with both E_1 and E_2, i.e. we want $E_1 ab \wedge E_2 ab$ to be satisfied in our model.

We use the standard description of a configuration: for each symbol a_i in the alphabet of M (including **blank**) we use the unary relation symbol A_i, for each state q_i - the unary symbol Q_i. We also have the unary symbol H describing the head position. An element x represents a tape cell scanned by the head if Hx and $Q_i x$, for some i, are true.

We begin our construction by enforcing that every model of φ contains a substructure that can be viewed as an infinite binary tree. The set of 2^{n^k} elements describing a single configuration of M is treated as a "node" of this tree.

We organize the structure in such a way that elements belonging to an even configuration, i.e. a configuration whose depth in the tree is even, are connected to the elements from the successor configurations by E_1, and elements belonging to an odd configuration are connected to the elements from the successor configurations by E_2. We do not impose any relations between elements that do not belong to a same configuration or to two consecutive configurations. Additionally, we introduce two unary symbols: D_0

true for elements belonging to even configurations, and D_1 true for elements belonging to odd configurations. One more unary symbol L indicates that the element belongs to the left son of some node.

Let us start the construction with stating the existence of the node representing the initial configuration of M on w. For elements of this node the special unary symbol I is true. We assume that for this configuration L is true:

$$\exists x\ (Ix \wedge D_0 x \wedge Lx \wedge \overline{P}(x) = 0, \tag{17}$$

$$\forall x (Ix \to (\overline{P}(x) \neq 2^{n^k} - 1 \to \exists y\ (E_1 xy \wedge E_2 xy \wedge Iy \wedge D_0 y \wedge Ly \wedge \overline{P}(y) = \overline{P}(x) + 1))). \tag{18}$$

The next formula expresses that for every element, except the last one, belonging to a description of a configuration there exists a successor in this configuration:

$$\bigwedge_{i=0,1} \forall x\ (D_i x \to (\overline{P}(x) \neq 2^{n^k} - 1$$
$$\to (\exists y\ (E_1 xy \wedge E_2 xy \wedge D_i x \wedge (Lx \leftrightarrow Ly) \wedge \overline{P}(y) = \overline{P}(x) + 1)))). \tag{19}$$

The existence of successor nodes is implied by the following formulas, in which $n \oplus m \equiv n + m$ (mod 2):

$$\bigwedge_{i=0,1} \forall x\ (D_i x \to \exists y\ (E_{i+1} xy \wedge D_{i \oplus 1} y \wedge Ly \wedge \overline{P}(y) = 0)), \tag{20}$$

$$\bigwedge_{i=0,1} \forall x\ (D_i x \to \exists y\ (E_{i+1} xy \wedge D_{i \oplus 1} y \wedge \neg Ly \wedge \overline{P}(y) = 0)). \tag{21}$$

Now we say that a model of our formula satisfies several basic properties of a computation tree. We can say that there is exactly one alphabet symbol in every tape cell:

$$\bigwedge_{k=0,1} \forall x\ (D_k x \to \bigvee_i A_i x) \quad \wedge \quad \bigwedge_j \forall x\ (A_j x \to \bigwedge_{i \neq j} \neg A_i x). \tag{22}$$

In each configuration at most one element is scanned by the head:

$$\forall xy\ (E_1 xy \to (E_2 xy \to ((Hx \wedge Hy) \to x = y))). \tag{23}$$

Exactly the elements which represent tape cells observed by the head store information about state.

$$\bigwedge_i \forall x\ (Q_i x \to Hx) \quad \wedge \quad \forall x\ (H(x) \to \bigvee_i Q_i x). \tag{24}$$

The next formulas ensure that the root of the tree describes the initial configuration in the state Q_0 of M on the input $w = a_0 a_1 \ldots a_{n-1}$.

$$\forall x\ (Ix \to (\overline{P}(x) = 0 \to (Hx \wedge Q_0 x))), \tag{25}$$

$$\bigwedge_{i<n} (\forall x\ (Ix \to (\overline{P}(x) = i \wedge \to W_i x))) \ \wedge \ \forall x\ (Ix \to (\overline{P}(x) \geq n \to Bx)), \tag{26}$$

where B is the symbol representing **blank**.

The following formula says that if a tape cell of a configuration is not scanned by the head then in the same cell of both successor configurations the alphabet symbol does not change.

$$\bigwedge_{i=0,1} \forall xy\ (E_{i+1}xy \rightarrow ((D_i x \wedge D_{i\oplus 1} y \wedge \neg Hx \wedge \overline{P}(x) = \overline{P}(y)) \rightarrow \bigwedge_i (A_i x \leftrightarrow A_i y))). \tag{27}$$

Consider now a node t of the tree and the configuration c that is described by this node. There are two cases: the state of the machine in this configuration is existential or it is universal.

In the first case we enforce that the configuration represented by the left son of t is created by applying one of the two possible transitions on c. Assume that for an existential state q and a letter a there are two possible transitions: $(q, a) \rightarrow (q', a', \rightarrow)$ and $(q, a) \rightarrow (q'', a'', \leftarrow)$. We put:

$$\begin{aligned} \bigwedge_{i=0,1} \forall xy \quad \big(E_{i+1}xy \rightarrow \big(((D_i x \wedge D_{i\oplus 1} y \wedge Qx \wedge Ax \wedge Ly \wedge \overline{P}(x) = \overline{P}(y)) \\ \rightarrow \big((A'y \wedge \forall x\ (E_1 xy \rightarrow (Next(y,x) \rightarrow Hx \wedge Q'x))) \\ \vee (A''y \wedge \forall x\ (E_1 xy \rightarrow (Next(x,y) \rightarrow Hx \wedge Q''x))))\big)\big)\big), \end{aligned} \tag{28}$$

where the formula $Next(x, y)$ is the abbreviation stating that x and y are two consecutive elements in the representation of a configuration: $Next(x, y) \equiv E_1 xy \wedge E_2 xy \wedge (D_0 x \wedge D_0 y \vee D_1 x \wedge D_1 y) \wedge \overline{P}(y) = \overline{P}(x) + 1$. Other possible situations, when both transitions move the head forward or both transitions move the head backward, can be handled similarly.

Consider now the case of a universal configuration. We enforce that the left son of t is created by applying the first transition and the right son by applying the second one. For a universal state q, a letter a and transitions $(q, a) \rightarrow (q', a', \rightarrow)$ and $(q, a) \rightarrow (q'', a'', \leftarrow)$ we put:

$$\begin{aligned} \bigwedge_{i=1,2} \forall xy \quad \big(\ E_{i+1}xy \rightarrow \big(((D_i x \wedge D_{i\oplus 1} y) \wedge\ Qx \wedge Ax \wedge \overline{P}(x) = \overline{P}(y)) \\ \rightarrow \big(\neg Ly \rightarrow \big(A''x \wedge \forall x\ (E_1 xy \rightarrow (Next(x,y) \rightarrow (Q''x \wedge Hx))))\big) \\ \wedge\ Ly \rightarrow \big(A'x \wedge \forall x\ (E_1 xy \rightarrow (Next(y,x) \rightarrow (Q'x \wedge Hx))))\big)\big)\big). \end{aligned} \tag{29}$$

To finish our construction we give a formula stating that the machine never enters the only rejecting state q_r.

$$\forall x\ (Q_r x \rightarrow \mathbf{false}). \tag{30}$$

Now, let φ be a conjunction of (17)–(30) (plus formulas symmetric to (28) and (29)). Observe that the number of conjuncts and the size of each of them are polynomial in the size of M and w.

We claim that φ is satisfiable if and only if M accepts w. Indeed, if M accepts w then an accepting computation tree can be transformed into a model $\mathfrak{M}$ of φ in the following way. The root of the computation tree is transformed into the root of $\mathfrak{M}$.

Then we proceed recursively. Let c be a configuration in the computation tree and let c' be its code in $\mathfrak{M}$. If c is universal then we transform its left subtree into the left subtree of c' and its right subtree into the right subtree of c'. If c is existential then we transform its accepting subtree into the left subtree of c'. Since we want to have a complete binary tree, we have to define somehow also the right subtree of c'. We can for example construct all nodes of this subtree in such a way that they agree with c' in predicates denoting alphabet symbols and for each element a from these nodes $\mathfrak{M} \models \neg Ha \wedge \bigwedge_i \neg Q_i a$. It is easy to see that $\mathfrak{M}$ is indeed a model of our formula.

For the proof of the opposite direction we want to check that the existence of an accepting computation tree is implied by the existence of a model $\mathfrak{M}$ of φ. The set of elements of $\mathfrak{M}$ whose existence is enforced by (17), (18) is translated into the root of computation tree. It describes the initial configuration of M on w because of (25), (26). The construction of the further parts of the computation tree is recursive. Let us assume that we have constructed a configuration c, which is encoded in $\mathfrak{M}$ by the node consisting of $c_1, \ldots, c_{2^{n^k}}$. Let us go for example through the case when c is an even configuration, i.e. for all i we have $\mathfrak{M} \models D_0 c_i$. Let a_0 and b_0 be the elements whose existence is ensured by (20), (21). We have $\mathfrak{M} \models D_1 a_0 \wedge D_1 b_0 \wedge La_0 \wedge \neg Lb_0$. By (19) there exist two nodes consisting of $a_0, \ldots a_{2^{n^k}}$ and $b_0, \ldots b_{2^{n^k}}$ connected to the node with c_i by E_1 such that:

- for all i: $\mathfrak{M} \models \bar{P}(a_i) = \bar{P}(b_i) = i$,
- for all i: $\mathfrak{M} \models La_i \wedge \neg Lb_i$,
- for $i < 2^{n^k} - 1$: $\mathfrak{M} \models a_i E_1 a_{i+1} \wedge a_i E_2 a_{i+1} \wedge b_i E_1 b_{i+1} \wedge b_i E_2 b_{i+1}$.

If c is universal then we translate the node $a_0, \ldots, a_{2^{n^k}}$ into the left successor of c and $b_0, \ldots, b_{2^{n^k}}$ into the right successor of c. Consider now the case of existential c. Because of the appropriate formula of the form (28) we can translate $a_0, \ldots, a_{2^{n^k}}$ into one of the successors of c. At this moment we leave the second successor undefined.

At the end, for formal conformity, we substitute undefined subtrees of existential nodes with subtrees which agree with transition function of M. This is not essential since in existential nodes we demand only one accepting successor. The construction of φ, in particular its conjuncts of type (28) and (29), implies that the tree obtained in the described way is indeed an accepting computation tree of $\mathcal{M}$ on w.

Remark. Note that the usage of equivalence relation symbols outside guards in our proof is very limited. Actually, we only need guards which are conjunctions of two atoms: $E_1 xy \wedge E_2 xy$, instead of just single atomic formulas.

The Upper Bound. In this proof we use the notion of *intersection*. An intersection (in a relational structure with E_1, E_2 equivalence relations) is an equivalence class of the relation $E_1 \cap E_2$. To denote intersections we use letters $\mathcal{I}$, $\mathcal{J}$. By $\Theta^{\mathfrak{A}}(\mathcal{I})$ we denote the isomorphism type of $\mathcal{I}$ in $\mathfrak{A}$. Equivalence classes of E_1 and E_2 are called E_1-classes and E_2-classes, respectively. Let φ be a satisfiable formula in $[E_1, E_2]$-guarded normal form. From an arbitrary model of φ we may obtain a model with all intersections of size bounded exponentially in $|\varphi|$. It was shown for FO^2+$equiv[E_1, E_2]$ in [11]. Let $\mathfrak{A} \models \varphi$ be a model of φ with so exponentially bounded intersections. We construct a new, forest-like model $\mathfrak{B} \models \varphi$. $\mathfrak{B}$ is built from isomorphic copies of intersections from $\mathfrak{A}$, thus we do not have to bother about witnesses for conjuncts of the form $\forall\exists^{++}$.

Several times in the construction we have the following task: for a given E_i-class $\mathcal{C}$ of $\mathfrak{A}$, with a distinguished intersection $\mathcal{I}$, build an E_i-class $\mathcal{D}$ of $\mathfrak{B}$, of size exponential in $|\varphi|$, such that $\mathcal{D}$ contains an isomorphic copy of $\mathcal{I}$ as an intersection. The desired $\mathcal{D}$ may be obtained in the same way as an exponential model $\mathfrak{M}$ of a satisfiable FO^2+$equiv[E]$ sentence ψ in [11]. The class $\mathcal{C}$ in our case corresponds to $\mathfrak{M}$ in the construction in [11], intersections of $\mathcal{C}$ correspond to E-classes of $\mathfrak{M}$. We skip details here.

The construction of $\mathfrak{B}$ is divided into (possibly) infinite number of stages:

Layer 0. For every conjunct of φ of the form $\exists x(\alpha(x) \wedge \chi(x))$ we find an element $a \in \mathfrak{A}$ such that $\mathfrak{A} \models \alpha(x) \wedge \chi(x)$. Let $a \in \mathcal{I} \subseteq \mathcal{C}$, where $\mathcal{C}$ is the E_1-class of a. We put a new intersection $\mathcal{J}$, such that $\Theta^{\mathfrak{A}}(\mathcal{I}) = \Theta^{\mathfrak{B}}(\mathcal{J})$, into the Layer 0 of $\mathfrak{B}$. We construct a regular, exponential-size version $\mathcal{D}$ of $\mathcal{C}$, containing $\mathcal{J}$ as an intersection. After building $\mathcal{D}$ we may assume that all its elements have all the required witnesses for conjuncts of the form $\forall\exists^{+-}$.

The Next Layers. We proceed recursively. Having constructed the Layer i we construct the Layer $i+1$. Let $\mathcal{J}$ be an intersection belonging to the i-th Layer. If i is even then the whole E_1-class of $\mathcal{J}$ is constructed. Similarly if i is odd then the whole E_2-class of $\mathcal{J}$ is constructed. Assume that i is even. The case of i odd is symmetric. We want to construct E_2-class of $\mathcal{J}$, thus providing all the witnesses for formulas of the type $\forall\exists^{-+}$. Let $\mathcal{I} \subseteq \mathfrak{A}$ be such that $\Theta^{\mathfrak{A}}(\mathcal{I}) = \Theta^{\mathfrak{B}}(\mathcal{J})$. Let $\mathcal{C}$ be the E_2-class containing $\mathcal{I}$. We construct a regular, exponential-size version $\mathcal{D}$ of $\mathcal{C}$, containing $\mathcal{J}$ as an intersection. We put all the intersections of $\mathcal{D}$, with the exception of $\mathcal{J}$ into the Layer $i+1$ of $\mathfrak{B}$. Note that it may happen that $\mathcal{D}$ consists only of $\mathcal{J}$ and then there is nothing to put into the next layer.

Now, let us explain how to provide witnesses for conjuncts of the type $\forall\exists^{--}$ for elements from $\mathcal{J}$. Let $b \in \mathcal{J}$. Consider a conjunct of φ of the form $\forall x\ (\alpha(x) \rightarrow \exists y\ (\beta(x,y) \wedge \neg E_1xy \wedge \neg E_2xy \wedge \chi(x,y)))$. Let $a \in \mathfrak{A}$ be such that $type^{\mathfrak{A}}(a) = type^{\mathfrak{B}}(b)$.[5] If $\mathfrak{A} \models \alpha(a)$ then find an element $a' \in \mathcal{I}' \subseteq \mathfrak{A}$ such that $\mathfrak{A} \models \beta(a,a') \wedge \neg E_1aa' \wedge \neg E_2aa' \wedge \chi(a,a)$. Put a new intersection $\mathcal{J}'$ into the Layer $i+1$. Set $\Theta^{\mathfrak{B}}(\mathcal{J}') = \Theta^{\mathfrak{A}}(\mathcal{I}')$. Let b' be an element of type $type^{\mathfrak{A}}(a')$ in $\mathcal{J}'$. Set $type^{\mathfrak{B}}(b,b') = type^{\mathfrak{A}}(a,a')$. Set all the other 2-types between elements from $\mathcal{J}'$ and the elements from the Layer i in such a way that they are *negative*, i.e. contain no positive occurrences of binary predicates. Extend $\mathcal{J}'$ to its E_2-class.

For all the pairs of elements for which we have not set their 2-type yet, we set them in such a way that they are negative. We can do so safely, without violating the conjuncts of φ of the form $\forall\forall$, because all of such conjuncts say something only about elements connected by binary relations.

This finishes the construction of a regular model. Observe that to check the existence of a regular model constructed in the described way it suffices to build doubly exponential number of levels of a model. It is enough because the number of isomorphism types of intersections which may appear is at most doubly exponential. So later we may just repeat the structure of trees. An alternating procedure working in exponential space, checking whether a GF^2+EQ[E_1, E_2] sentence in [E_1, E_2]-normal form has a regular model, can be naturally derived from the construction. $\square$

[5] $type^{\mathfrak{A}}(a)$ denotes the 1-type realized by a in $\mathfrak{A}$.

6 Loosely Guarded Fragment with One Equivalence/Transitive Relation

The guarded fragment with equivalence relations GF+$equiv$ and the guarded fragment with transitive relations GF+$trans$ are undecidable. We know, however, that if we restrict the usage of equivalence/transitive relations to guards only then the obtained versions: GF+EG and GF+TG become decidable. In this section we show that this result cannot be generalized to the loosely guarded fragment LGF. We prove that allowing just one equivalence or transitive relation makes LGF undecidable, even if this special relation is used only in guards.

First we give an easy proof of:

Theorem 6. *LGF*^{3}*+*$TG[T]$ *forms a conservative reduction class.*

Proof. Let φ be the conjunction of (31)-(34) below and the transitivity axiom $trans[T]$.

$$\exists x\ Nx, \tag{31}$$

$$\forall x\ (Nx \to \exists y\ (Txy \wedge Hxy \wedge Ny)), \tag{32}$$

$$\forall x\ (Nx \to \exists y\ (Tyx \wedge Vxy \wedge Ny)), \tag{33}$$

$$\forall xyz((Vyx \wedge Hyz \wedge Txz) \to \exists y(Txz \wedge Txy \wedge Tyz \wedge Hxy \wedge Vzy \wedge Ny)). \tag{34}$$

It is very easy to expand $\mathfrak{G}_{\mathbb{N}}$ to a model of φ. We make N true for all elements and set T to be the transitive closure of $\{((x_1, y_1), (x_2, y_2)) : x_2 = x_1 + 1 \wedge y_1 = y_2 \vee y_1 = y_2 + 1 \wedge x_1 = x_2\}$. Similarly, an expansion of $\mathfrak{G}_m$ to a model of φ is obtained by setting T to be the transitive closure of $\{((x_1, y_1), (x_2, y_2)) : 0 \leq x_i, y_i < m, x_2 = x_1 + 1 \pmod m \wedge y_1 = y_2 \vee y_1 = y_2 + 1 \pmod m\ \wedge x_1 = x_2\}$.

Let us consider an infinite model $\mathfrak{A} \models \varphi$. We argue that $\mathfrak{A}' = \mathfrak{A}{\restriction}N$ is grid-like, i.e. we construct a homomorphism $h : \mathfrak{G}_{\mathbb{N}} \to \mathfrak{A}'$. We start with embedding $\{(x, 0) : x \in \mathbb{N}\}$ into $\mathfrak{A}'$. Choose an element a_0 in $\mathfrak{A}'$ satisfying (31) and set $h(0, 0) = a_0$. Now if $h(x, 0) = a_x$ then let $h(x + 1, 0)$ be the witness for a_x and (32). Assume now that we have defined $h(x, y)$ for all $x \in \mathbb{N}$ and $y < k$. We say how to extend h to the $k - th$ row of $\mathfrak{G}_{\mathbb{N}}$. Choose $h(0, k)$ as a witness for $h(0, k - 1)$ and (33). Now, let $h(1, k)$ be a witness for $h(0, k)$, $h(0, k-1)$ and $h(1, k-1)$ and (34). We define $h(x, k)$ in sequence as witnesses for $h(x - 1, k)$, $h(x - 1, k - 1)$, $h(x, k - 1)$ and (34).

We skip here the argumentation for grid-likeness of finite models. It goes the same way as the proof of Lemma 2.4 in [13]. □

Observe that if we exchange the transitivity axiom $trans[T]$ in the above proof by $equiv[T]$ then the proof still works correctly. Thus we have:

Theorem 7. *LGF*^{3}*+*$EG[T]$ *forms a conservative reduction class.*

Remark. If we want to obtain only undecidability of LGF^3+$TG[T]$ rather then the fact that it forms a conservative reduction class, we can make it even in the case when T is the only non-unary symbol allowed. In this case we use the formula stating that between two elements connected by T there is no other element:

$$NB(x, y) \equiv \neg\exists z\ (Txy \wedge Txz \wedge Tzy).$$

Now we can simulate Hxy and Vxy by $NB(x, y)$ and values of some additional unary predicates (in particular the predicate D which is true exactly for elements in even rows). For example the formula (34) should be substituted by

$$\begin{aligned}\forall xyz \;\; & ((Txy \wedge Tyz \wedge Txz) \rightarrow (((NB(x, y) \wedge NB(y, z) \\ & \wedge(Dx \wedge \neg Dy \wedge \neg Dz \vee \neg Dx \wedge Dy \wedge Dz)) \\ & \rightarrow \exists y(Txy \wedge Tyz \wedge Txz \wedge NB(x, y) \wedge NB(y, z) \wedge (Dx \leftrightarrow Dy)))) \quad (35)\end{aligned}$$

Note that due to the lack of the orientation of E this trick does not work for LGF^3+ $EG[E]$.

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