Finite Satisfiability of Modal Logic over Horn Definable Classes of Frames

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Abstract

Modal logic plays an important role in various areas of computer science, including verification and knowledge representation. In many practical applications it is natural to consider some restrictions of classes of admissible frames. Traditionally classes of frames are defined by modal axioms. However, many important classes of frames, e.g. the class of transitive frames or the class of Euclidean frames, can be defined in a more natural way by first-order formulas. In a recent paper it was proved that the satisfiability problem for modal logic over the class of frames defined by a universally quantified, first-order Horn formula is decidable. In this paper we show that also the finite satisfiability problem for modal logic over such classes is decidable.

Keywords: modal logic, decidability, finite satisfiability

1 Introduction

Modal logic was introduced by philosophers to study modes of truth. The idea was to extend propositional logic by some new constructions, of which two most important were $\Diamond\varphi$ and $\Box\varphi$, originally read as φ *is possible* and φ *is necessary*, respectively. A typical question was, given a set of axioms $\mathcal{A}$, corresponding usually to some intuitively acceptable aspects of truth, what is the logic defined by $\mathcal{A}$, i.e. which formulas are provable from $\mathcal{A}$ in a Hilbert-style system.

One of the most important steps in the history of modal logic was the invention of a formal semantics based on the notion of the so-called Kripke structures. Basically, a Kripke structure is a directed graph, called a *frame*, together with a valuation of propositional variables. Vertices of this graph are called *worlds*. For each world truth values of all propositional variables can be defined independently. In this semantics, $\Diamond\varphi$ means *the current world is connected to some world in which* φ *is true*; and $\Box\varphi$, equivalent to $\neg\Diamond\neg\varphi$, means φ *is true in all worlds to which the current world is connected.*

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It appeared that there is a beautiful connection between syntactic and semantic approaches to modal logic [21]: logics defined by axioms can be often equivalently defined by restricting classes of frames. E.g., the axiom $\Diamond\Diamond P \Rightarrow \Diamond P$ (*if it is possible that P is possible, then P is possible*), is valid precisely in the class of transitive frames; the axiom $P \Rightarrow \Diamond P$ (*if P is true, then P is possible*) – in the class of reflexive frames, $P \Rightarrow \Box\Diamond P$ (*if P is true, then it is necessary that P is possible*) – in the class of symmetric frames, and the axiom $\Diamond P \Rightarrow \Box\Diamond P$ (*if P is possible, then it is necessary that P is possible*) – in the class of Euclidean frames.

Many important classes of frames, in particular all the classes we mentioned above, can be defined by simple first-order formulas. For a given first-order sentence Φ over the signature consisting of a single binary symbol R we define $\mathcal{K}_\Phi$ to be the set of those frames which satisfy Φ.

Decidability over various classes of frames can be shown by employing the so-called standard translation of modal logic to first-order logic. Indeed, the satisfiability of a modal formula φ in $\mathcal{K}_\Phi$ is equivalent to satisfiability of $st(\varphi) \wedge \Phi$, where $st(\varphi)$ is the standard translation of φ. In this way, we can show that (multi)modal logic is decidable over any class defined by two-variable logic [17], even extended with a linear order [18], counting quantifiers [19,5,20], one transitive relation [22], two equivalence relations [12,14], or equivalence closures of two distinguished binary relations [11]. The same holds for formulas of the guarded fragment [4], even if we allow for some restricted application of transitive relations [23,13], fixed-points [6,1] and transitive closures [15]. In many cases the decidability results hold also when only finite frames are considered. The complexity bounds obtained this way, however, are high — usually between ExpTime and 2NExpTime.

Clearly, some modal logics defined by a first-order formula are undecidable. A stronger result was presented in [7] — it was shown that there exists a universal first-order formula with equality such that the global satisfiability problem over the class of frames that satisfy this formula is undecidable. In [9], this result was improved — it was shown that equality is not necessary. The proof from [9] works also for local satisfiability. Finally, in [10] it was shown that even a very simple formula with three variables without equality may lead to undecidability.

The classical classes of frames we mentioned earlier, i.e. transitive, reflexive, symmetric and Euclidean are decidable. They can be defined by first-order sentences even if we further restrict the language to universal Horn formulas without equality, UHF. Universal Horn formulas were considered in [8], where a dichotomy result was proved, that the satisfiability problem for modal logic over the class of frames defined by a UHF formula (with an arbitrary number of variables) is either in NP or PSpace-hard. The authors of [8] conjectured that the problem is decidable in PSpace for all universal Horn formulas. This conjecture was confirmed in [16].

In case of some UHF formulas Φ, decidability of corresponding modal logics is shown in [16] by demonstrating the finite model property with respect to $\mathcal{K}_\Phi$, i.e. by proving that every modal formula satisfiable over K_Φ has also a finite model in K_Φ. However, it is not always possible, as it is not hard to construct a UHF formula Φ, such that some modal formulas have only infinite models over K_Φ. Assume e.g that Φ enforces irreflexivity and transitivity, and consider the following modal formula: $\Diamond p \wedge \Box\Diamond p$.

This naturally leads to the question, whether for any UHF formula Φ the finite satisfiability problem for modal logic over K_Φ is decidable. This question is particularly important, if one considers practical applications, in which the structures (corresponding e.g. to knowledge bases or descriptions of programs) are usually required to be finite.

Decision procedures for the finite satisfiability problem for modal and related logics are very often more complex than procedures for general satisfiability. As argued in [25], the model theoretic reason for the good behavior of modal logics is the tree model property. A standard technique is to unravel an arbitrary model into a (usually infinite) tree. In [16] we also apply this idea (at least as a starting point of our constructions, as the obtained unravellings have to be sometimes modified to meet the requirements of the UHF formula defining the class of frames). Clearly such an approach, is not sufficient if we are interested only in finite models.

In this paper we are however able to positively answer the given question:

Theorem 1.1 *Let Φ be a universal Horn formula. Then the finite local and the finite global satisfiability problems for modal logic over $\mathcal{K}_\Phi$ are decidable.*

The precise statement of the results, containing also some complexity bounds, is given in Table 1.

Plan of the paper In Section 2 we define all important notions related to modal logic and Horn formulas, and then recall some definitions and results from [16]. The remaining part of the paper is divided into two sections each of which deals with one subclass of universal Horn formulas.

2 Preliminaries

2.1 Modal logic

As we work with both first-order logic and modal logic we help the reader to distinguish them in our notation: we denote first-order formulas with Greek capital letters, and modal formulas with Greek small letters.

We assume that the reader is familiar with first-order logic and propositional logic. Modal logic extends propositional logic with the $\Diamond$ operator and its dual $\Box$. Formulas of modal logic are interpreted in Kripke structures, which are triples of the form $\langle W, R, \pi\rangle$, where W is a set of elements, called *worlds*, $\langle W, R\rangle$ is a directed graph called a *frame*, and π is a function that assigns to each world a set of propositional variables which are true at this world. We say that a structure $\langle W, R, \pi\rangle$ is *based* on the frame $\langle W, R\rangle$. For a given class of frames $\mathcal{K}$, we say that a structure is $\mathcal{K}$-based if it is based on some frame from $\mathcal{K}$. We will use calligraphic letters $\mathcal{M}, \mathcal{N}$ to denote frames and Fraktur letters $\mathfrak{M}, \mathfrak{N}$ to denote structures.

For a frame $\langle W, R\rangle$ and a subset $W' \subseteq W$, we define $R_{\restriction W'} = R \cap (W' \times W')$. Similarly, for a labeling function π, we define $\pi_{\restriction W'}$ to be such that $\pi_{\restriction W'}(w) = \pi(w)$ for all $w \in W'$. We define the restriction of a frame $\langle W, R\rangle_{\restriction W'}$ to $W' \subseteq W$ as $\langle W', R_{\restriction W'}\rangle$.

The semantics of modal logic is defined recursively. A modal formula φ is (locally) *satisfied* in a world w of a model $\mathfrak{M} = \langle W, R, \pi\rangle$, denoted as $\mathfrak{M}, w \models \varphi$, if

(i) $\varphi = p$ where p is a variable and $\varphi \in \pi(w)$,

(ii) $\varphi = \neg\varphi'$ and this is not the case that $\mathfrak{M}, w \models \varphi'$,

(iii) $\varphi = \varphi_1 \vee \varphi_2$ and $\mathfrak{M}, w \models \varphi_1$ or $\mathfrak{M}, w \models \varphi_2$,

(iv) $\varphi = \Diamond\varphi'$ and there exists a world $v \in W$ such that $(w, v) \in R$ and $\mathfrak{M}, v \models \varphi'$,

The $\Box$ operator is dual to $\Diamond$: $\Box\varphi \equiv \neg\Diamond\neg\varphi$. Other logical connectives are defined in a standard way.

We say that a formula φ is *globally* satisfied in $\mathfrak{M}$, denoted $\mathfrak{M} \models \varphi$, if for all worlds w of $\mathfrak{M}$, we have $\mathfrak{M}, w \models \varphi$.

We consider satisfiability of modal formulas in restricted classes of frames. For a given class of frames $\mathcal{K}$ we say that a modal formula φ is *locally (globally) satisfiable over* $\mathcal{K}$ if there exists a $\mathcal{K}$-based structure $\mathfrak{M}$ and a world w of $\mathfrak{M}$ such that $\mathfrak{M}, w \models \varphi$ (resp. $\mathfrak{M} \models \varphi$). We are particularly interested in finite models. We say that a modal formula φ is *finitely locally (globally) satisfiable over* $\mathcal{K}$ if there exists a *finite* $\mathcal{K}$-based structure $\mathfrak{M}$ and a world w of $\mathfrak{M}$ such that $\mathfrak{M}, w \models \varphi$ (resp. $\mathfrak{M} \models \varphi$).

We define *local, global, finite local*, and *finite global satisfiability problem* for modal logic *over* $\mathcal{K}$ ($\mathcal{K}$-SAT, global-$\mathcal{K}$-SAT, $\mathcal{K}$-FINSAT, global-$\mathcal{K}$-FINSAT) as the question whether a given modal formula φ is locally, globally, finitely locally, resp. finitely globally satisfiable over $\mathcal{K}$.

We say that modal logic has the *finite model property* (resp. *finite global model property*), FMP, *with respect to a class of frames* $\mathcal{K}$, if any formula that is locally (resp. globally) satisfiable over $\mathcal{K}$ is also finitely locally (resp. globally) satisfiable over $\mathcal{K}$.

In our constructions we use the following terminology. A world w is *k-followed* (*k-preceded*) in a frame $\mathcal{M}$, if there exists a directed path $(w, u_1, u_2, \ldots, u_k)$ (resp. $(u_1, u_2, \ldots, u_k, w)$) in $\mathcal{M}$. Note that we do not require this path to consist of distinct elements. We say that a world w is *k-inner* in $\mathcal{M}$ if it is k-preceded and k-followed. We use also naturally defined notions of ∞-preceded, ∞-followed, and ∞-inner worlds. In particular, a world on a cycle is ∞-inner.

We employ a standard notion of a type. For a given formula φ, a Kripke structure $\mathfrak{M}$, and a world $w \in W$ we define the *type* of w (with respect to φ) in $\mathfrak{M}$ as $tp^{\varphi}_{\mathfrak{M}}(w) = \{\psi : \mathfrak{M}, w \models \psi$ and ψ is subformula of $\varphi\}$. We write $tp_{\mathfrak{M}}(w)$ if the formula is clear from context. Note that $|tp^{\varphi}_{\mathfrak{M}}(w)| \leq |\varphi|$, where $|\varphi|$ denotes the length of φ.

2.2 Universal Horn formulas

We use universal Horn formulas to define classes of frames. A Horn *clause* is a disjunctions of literals of which at most one is positive. The set of *universal Horn formulas*, UHF, is defined as the set of those Φ over the language $\{R\}$ (without equality) which are of the form $\forall \boldsymbol{x}.\Phi_1 \wedge \Phi_2 \wedge ... \wedge \Phi_n$, where each Φ_i is a Horn clause. We usually present Horn clauses as implications and skip the quantifiers. For example, the UHF formula $(xRy \wedge yRz \Rightarrow xRz) \wedge (xRx \Rightarrow \bot)$ defines the set of transitive and irreflexive frames. We assume without loss of generality that each Horn clause consists of variables x, y, $\ldots$, $z_1, z_2, \ldots$, and is of the form $\Psi \Rightarrow \bot$, $\Psi \Rightarrow xRx$, or $\Psi \Rightarrow xRy$. We define $\Psi(v_x, v_y, v_1, \ldots, v_k)$ as the instantiation of Ψ with $x = v_x$, $y = v_y$, $z_1 = v_1$, $z_2 = v_2, \ldots$, e.g. $(xRz_1 \wedge z_1Rz_2 \wedge z_2Ry \Rightarrow xRy)(a, b, c, d) = aRc \wedge cRd \wedge dRb \Rightarrow aRb$.

For a given $\Phi \in \mathsf{UHF}$, we write $\mathcal{K}_\Phi$ for the class of frames satisfying Φ. When considering $\mathcal{K}_\Phi$-SAT, global $\mathcal{K}_\Phi$-SAT, and their finite versions, the formula Φ is fixed and is not a part of the input. However, the complexity depends on this formula. To hide unnecessary details, we use a function $\mathfrak{g}$ to bound the size of models or complexity in the size of Φ. Please keep in mind that once Φ is fixed, $\mathfrak{g}(|\Phi|)$ can be treated as a constant, and therefore the precise value of $\mathfrak{g}$ is not important (see also [16]).

2.3 Consequences, closures and morphisms

In this and in the next subsection we recall some notions and results from [16]. Observations related to the content of this subsection appear also in [3].

We say that an edge (w, w') is a *consequence* of Φ in $\mathcal{M} = \langle W, R\rangle$, if for some worlds $v_1, \ldots, v_k \in W$ and a clause $\Psi_1 \Rightarrow \Psi_2$ of Φ we have $\mathcal{M} \models \Psi_1(w, w', v_1, \ldots, v_k)$, and $\Psi_2(w, w', v_1, \ldots, v_k) = wRw'$. We denote the set of all consequences of Φ in $\mathcal{M}$ by $\mathrm{C}^{\Phi}_{\leadsto}(\mathcal{M})$. We define the *consequence operator* as follows.

$$\textsc{Cons}_{\Phi,W}(R) = R \cup \mathrm{C}^{\Phi}_{\leadsto}(\langle W, R\rangle).$$

Now, the *closure operator* can be defined as the least fixed-point of *Cons*:

$$\textsc{Closure}_{\Phi,W}(R) = \bigcup_{i>0} \textsc{Cons}^{i}_{\Phi,W}(R).$$

For a given frame $\mathcal{M} = \langle W, R\rangle$ we denote by $\mathfrak{C}_\Phi(\mathcal{M})$ the frame $\langle W, \textsc{Closure}_{\Phi,W}(R)\rangle$. The following lemma says that when considering satisfiability over arbitrary models from $\mathcal{K}_\Phi$ we can restrict our attention to models which are closures of trees (note however that those trees are usually infinite).

Lemma 2.1 *Let φ be a modal formula and let $\Phi \in \mathsf{UHF}$. If φ is $\mathcal{K}_\Phi$-satisfiable, then there exists a tree $\mathcal{T}$ with the degree bounded by $|\varphi|$, and a labeling π, such that*

(i) $\langle \mathcal{T}, \pi_\mathcal{T}\rangle$ is a model of φ;

(ii) $\langle \mathfrak{C}_\Phi(\mathcal{T}), \pi_\mathcal{T}\rangle$ is a model of φ that satisfies Φ.

The result holds for local satisfiability and for global satisfiability.

In this paper we make a high use of morphisms. We say that a function f from a frame $\mathcal{M}_1$ into a frame $\mathcal{M}_2$ is a *morphism* iff for all worlds w, w' if $\mathcal{M}_1 \models wRw'$, then $\mathcal{M}_2 \models f(w)Rf(w')$. The following observation is straightforward.

Observation 2.2 *Let $\mathcal{M}_1, \mathcal{M}_2$ be frames, let $\Phi \in \mathsf{UHF}$ and let f be a function from $\mathcal{M}_1$ into $\mathcal{M}_2$. If f is a morphism from $\mathcal{M}_1$ into $\mathcal{M}_2$, then f is a morphism from $\mathfrak{C}_\Phi(\mathcal{M}_1)$ into $\mathfrak{C}_\Phi(\mathcal{M}_2)$.*

2.4 Properties of formulas, frames and models

In our analysis an important role is played by two simple frames. The linear frame $\mathcal{L}_\mathbb{Z}$, which is defined as $\langle \{\underline{i} : i \in \mathbb{Z}\}, \{(\underline{i}, \underline{i+1}) | i \in \mathbb{Z}\}\rangle$, and the infinite binary tree $\mathcal{T}_\infty$, defined as $\langle \{\underline{s} | s \in \{0,1\}^*\}, \{(\underline{s}, \underline{si}) | s \in \{0,1\}^* \wedge i \in \{0,1\}\}\rangle$. For each $s \in \mathbb{N}$, we define $\mathcal{I}_s = \mathcal{L}_{\mathbb{Z}\restriction W_s}$, where $W_s = \{\underline{i} | 0 \leq i < s\}$. Note that the frame $\mathcal{L}_\mathbb{Z}$ is based on the integers, so for any k, the shift function $sh_k(\underline{i}) = \underline{i+k}$ is an automorphisms of $\mathcal{L}_\mathbb{Z}$.

For an arbitrary tree $\mathcal{T}$, we define a morphism $h_\mathcal{T} : \mathcal{T} \to \mathcal{L}_\mathbb{Z}$ in such a way that for each v at the ith level of $\mathcal{T}$, $h_\mathcal{T}(v) = \underline{i}$.

We call a formula $\Phi \in \mathsf{UHF}$ *bounded* if $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ is not a model of Φ, and *unbounded* otherwise.

We say that a formula $\Phi \in \mathsf{UHF}$ *forks at level* i if for all $\underline{s} \in \mathcal{T}_\infty$ with $|s| = i$ and $t, t' \in \{0,1\}^*$ there is no edge between $\underline{s0t}$ and $\underline{s1t'}$ in $\mathfrak{C}_\Phi(\mathcal{T}_\infty)$.

We say that an edge $(\underline{i}, \underline{j})$ is *forward* if $i < j$, *backward* if $i > j$, *short* if $|i - j| < 2$, and *long* if $|i - j| \geq 2$. We say that Φ *forces* long (resp. backward) edges if there is a long (resp. backward) edge in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ and that Φ *forces only long forward edges* if it forces long edges but it does not force backward edges.

We say that $\Phi \in \mathsf{UHF}$ satisfies

S1 if Φ does not force long edges,

S2 if Φ forces only long forward edges and there exist $l, a_1, a_2, \ldots, a_l$ bounded by $\mathfrak{g}(|\Phi|)$ such that for all worlds $\underline{i}$, $\underline{i+b}$, there is an edge from $\underline{i}$ to $\underline{i+b}$ in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ if and only if $b \geq 0$ and $b-1$ is in the additive closure of $\{a_1, a_2, \ldots, a_l\}$.

S3 if Φ forces long and backward edges and there exists m bounded by $\mathfrak{g}(|\Phi|)$ such that for all worlds $\underline{i}$, $\underline{i+b}$, there is an edge from $\underline{i}$ to $\underline{i+b}$ in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ if and only if m divides $|b-1|$.

It appears that the above subcases cover all possibilities.

Lemma 2.3 *Each $\Phi \in \mathsf{UHF}$ satisfies S1, S2, or S3.*

Lemma 2.4 *If $\Phi \in \mathsf{UHF}$ is bounded, then each formula satisfiable over $\mathcal{K}_\Phi$ has a model over $\mathcal{K}_\Phi$ with polynomially many worlds. The result holds for local satisfiability and for global satisfiability.*

Note that the above lemma implies that global $\mathcal{K}_\Phi$-FINSAT and global $\mathcal{K}_\Phi$-SAT are NP-complete for every consistent and bouned Φ. A similar bound on the size of models was also shown for all formulas satisfying S3:

Properties of Φ	global-$\mathcal{K}_\Phi$-FINSAT	$\mathcal{K}_\Phi$-FINSAT
inconsistent	P (TRIVIAL)	
consistent and bounded	FMP, NP-c (2.4)	
unbounded, satisfies S2	NEXPTIME (4.2)	
unbounded, satisfies S3	FMP, NP-c [16]	
is unbounded, satisfies S1 and …		
…forks at all levels and merges at some level	Lack of FMP (3.3), PSPACE-c(3.8, 3.9)	FMP (3.1), PSPACE-c [16]
…forks at all levels and does not merge at any level	FMP (3.10), EXPTIME-c [16]	FMP (3.1), PSPACE-c [16]
…does not fork at some level	FMP (3.11), PSPACE-c [16]	FMP (3.1), NP-c [16]

Table 1
A summary of results for finite satisfiability of modal logic over classes of frames defined by Horn formulas.

Lemma 2.5 *If $\Phi \in \mathsf{UHF}$ satisfies S3 then each formula satisfiable over $\mathcal{K}_\Phi$ has a model over $\mathcal{K}_\Phi$ with polynomially many worlds. The result holds for local satisfiability and for global satisfiability.*

Thus, in this paper it remains to investigate unbounded formulas satisfying S1 (Section 3) or S2 (Section 4). Two further technical lemmas will be helpful.

Lemma 2.6 *Let $\Phi \in \mathsf{UHF}$, $\mathcal{T}$ be a tree and v_i, v_j be $\mathfrak{g}(|\Phi|)$-inner worlds at the same path. Then there is an edge from v_i to v_j in $\mathfrak{C}_\Phi(\mathcal{T})$ if and only if there is an edge from $h_\mathcal{T}(v_i)$ to $h_\mathcal{T}(v_j)$ in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$.*

We say that two worlds w, w' of a frame $\mathcal{M}$ are *equivalent* if for each world u we have uRw iff uRw'.

Lemma 2.7 *Let $\Phi \in \mathsf{UHF}$ be a formula that does not fork, $\mathcal{T}$ be a tree with a bounded degree and w be a world at level $\mathfrak{g}(|\Phi|)$ in $\mathfrak{C}_\Phi(\mathcal{T})$. Then for $n = 2\mathfrak{g}(|\Phi|) + 1$ and all i, all the n-followed descendants of w at level $n + i$ are equivalent in the frame $\mathfrak{C}_\Phi(\mathcal{T})$.*

3 Formulas that do not force long edges

In this section, we consider unbounded formulas $\Phi \in \mathsf{UHF}$ that satisfy S1. In the case of local satisfiability we show the finite model property, essentially by an application of a standard selection argument. The case of global satisfiability is much more complicated. In particular for some formulas Φ we have to deal with infinite models.

3.1 Local satisfiability

Proposition 3.1 *Let Φ be an unbounded UHF formula that does not force long edges. Then modal logic has the finite model property with respect to $\mathcal{K}_\Phi$.*

Proof. Assume that φ is locally $\mathcal{K}_\Phi$-satisfiable. Let $\mathcal{T}$ be a tree guaranteed by Lemma 2.1. Thus, there exists a model $\mathfrak{M}$ based on the frame $\mathfrak{C}_\Phi(\mathcal{T}) \in \mathcal{K}_\Phi$, such that $\mathfrak{M}, w \models \varphi$, where w is the root of $\mathcal{T}$. Recall the morphism $h_\mathcal{T} : \mathcal{T} \to \mathcal{L}_\mathbb{Z}$. By observation 2.2, $h_\mathcal{T}$ is also a morphism from $\mathfrak{C}_\Phi(\mathcal{T})$ to $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$. Since Φ does not force long edges, this implies that $\mathfrak{C}_\Phi(\mathcal{T})$ can only contain edges between nodes on the same level or on two consecutive levels.

In order to obtain a finite model, we simply remove from $\mathfrak{M}$ all worlds from levels greater than $|\varphi|$. Since the truth of φ depends only on the worlds that are reachable from the root w by a path whose length is bounded by $|\varphi|$ (more precisely: by the modal depth of φ), the resulting model is a finite model of φ and, of course, it satisfies Φ since Φ is universal. $\square$

We showed that φ has a $\mathcal{K}_\Phi$-based model if and only if it has a finite $\mathcal{K}_\Phi$-based model, so $\mathcal{K}_\Phi$-FINSAT coincides with $\mathcal{K}_\Phi$-SAT, which was proved in [16] to be PSpace-complete.

3.2 Global satisfiability

In the case of general satisfiability [16], it was enough to consider the behavior of a first order formula on $\mathcal{T}_\infty$ and $\mathcal{L}_\mathbb{Z}$. In the case of finite satisfiability, we need one more frame, which we call $\mathcal{X}$, that contains a world with in-degree 2.

Formally, we define the frame $\mathcal{X}$ as $\langle W_X, R_X \rangle$, where $W_X = \{\underline{i} | i \in \mathbb{Z}\} \cup \{\overline{i} | i \in \mathbb{Z} \setminus \{0\}\}$ and $R_X = \{(\underline{i}, \underline{i+1}) | i \in \mathbb{Z}\} \cup \{(\overline{i}, \overline{i+1}) | i \in \mathbb{Z} \setminus \{-1, 0\}\} \cup \{(\overline{-1}, \underline{0}), (\underline{0}, \overline{1})\}$. Fig. 1 shows a fragment of $\mathcal{X}$.

We say that a formula Φ *merges* at a level $k < 0$ if in $\mathfrak{C}_\Phi(\mathcal{X})$ there is an edge from $\underline{k-1}$ to $\overline{k}$. For example, the formula $\Phi = xRz \wedge zRv \wedge yRv \Rightarrow xRy$ merges. Note that $\mathcal{T}_\infty$ and $\mathcal{L}_\mathbb{Z}$ satisfy Φ.

We consider three cases. For each formula Φ of UHF such that Φ does not force long edges, merges at some level and forks at all levels, we show that modal logic does not have the finite global model property with respect to $\mathcal{K}_\Phi$ (Proposition 3.3), and that global $\mathcal{K}_\Phi$-FINSAT is PSpace–complete (Propositions 3.8 and 3.9). For the cases of formulas that do not force long edges, do not merge at any level and fork at all levels, and formulas that do not force long edges and do not fork at all levels, the decidability follows from the finite model property (Propositions 3.10 and 3.11).

Formulas that merge. The following lemma shows an important regularity in models of formulas that merge.

Lemma 3.2 *Let Φ be an unbounded* UHF *formula that does not force long edges, and merges at a level k, $\mathfrak{M}$ be a model of Φ, $v_1, v_2, \ldots, v_l$ be a walk (i.e. a path, but not necessarily simple) in $\mathfrak{M}$ such that all v_i are ∞-inner.*

(i) If $v_l R v_{l-c}$ for some $c > 0$, then for all $i > c$, $v_i R v_{i-c}$.

(ii) If $v_{l-c} R v_l$ for some $c > 0$, then for all $i > c$, $v_{i-c} R v_i$.

Proof. Let $\ldots, v_{-2}, v_{-1}, v_0, v_1$ and $v_l, v_{l+1}, \ldots$ be infinite walks in $\mathfrak{M}$. Such walks exist since v_1 and v_l are ∞-inner.

We prove (i) by induction. Assume that for some $i > 0$, for all $j > i$ we

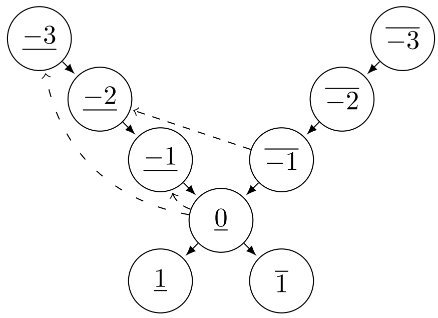

Fig. 1. A fragment of the frame $\mathcal{X}$ (circles and solid arrows). Consider a formula $\Phi = yRw \wedge wRv \wedge xRv \Rightarrow xRy$ that forces edge $(\overline{-1}, \underline{-2})$. When applied to $x = w = \underline{0}$, $y = \underline{-1}$ and $v = \underline{1}$, it forces edge $(\underline{0}, \underline{-1})$. Then, applied to $x = \underline{0}$, $v = \underline{-1}$, $w = \underline{-2}$ and $y = \underline{-3}$ it forces long edge $(\underline{0}, \underline{-3})$.

have $v_j R v_{j-c}$. We define a morphism h from $\mathcal{X}$ into $\mathfrak{M}$ as follows

$$h(w) = \begin{cases} v_{i+s+1} & \text{if } w = \underline{k+s} \text{ for some } s \leq 0 \\ v_{i-c+s} & \text{if } w = \underline{k+s} \text{ for some } s > 0 \\ v_{i-c+s} & \text{if } w = \overline{k+s} \text{ for some } s \in \mathbb{Z} \end{cases}$$

A quick check shows that h is indeed a morphism and since $\mathfrak{C}_\Phi(\mathcal{X})$ contains an edge from $\underline{k-1}$ to $\overline{k}$, $\mathfrak{M}$ has to contain edge from v_i to v_{i-c}.

The proof of (ii) is similar and thus omitted. □

Now we use the above lemma to show the lack of the finite model property.

Proposition 3.3 *Let Φ be an unbounded* UHF *formula that does not force long edges, merges at a level $k < 0$ and forks at all levels. Then modal logic lacks the finite global model property with respect to $\mathcal{K}_\Phi$.*

Proof. Let $\lambda = \bigwedge_{i \in \{0,1,2,3,4\}} \lambda_i$, where:

$$\begin{aligned} \lambda_0 &= \bigvee_{i \in \{1,2,3,4\}} p_i \;\wedge \bigwedge_{i,j \in \{1,2,3,4\}, i \neq j} \neg(p_i \wedge p_j) \\ \lambda_1 &= p_1 \Rightarrow (\Diamond p_2 \wedge \Box p_2) \\ \lambda_2 &= p_2 \Rightarrow (\Diamond p_3 \wedge \Box p_3) \\ \lambda_3 &= p_3 \Rightarrow (\Diamond p_2 \wedge \Diamond p_4 \wedge \Box(p_2 \vee p_4)) \\ \lambda_4 &= p_4 \Rightarrow (\Diamond p_1 \wedge \Box p_1) \end{aligned}$$

An infinite model of λ is presented in Fig. 2. It is not hard to see that its frame belongs to $\mathcal{K}_\Phi$ for any Φ meeting the assumptions.

Assume that $\mathfrak{M}$ is a finite $\mathcal{K}_\Phi$-based model of λ and let w be a world that satisfies p_1 in $\mathfrak{M}$. Quick check shows that such a world must exist. Let $w, w_1, w_2, \ldots$ be an infinite path in $\mathfrak{M}$ such that for odd i, w_i satisfies p_2, and for even i, w_i satisfies p_3. Such a path is guaranteed by λ_1, λ_2 and λ_3.

Since $\mathfrak{M}$ is finite, it must be the case that for some $0 < r < l$ we have $\mathfrak{M} \models w_l R w_r$. Clearly, w_l and w_k are ∞-inner. It follows from Lemma 3.2 that

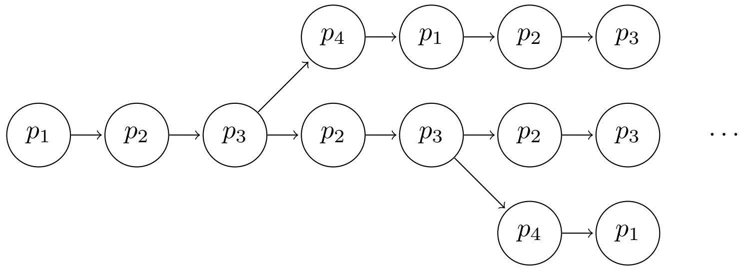

Fig. 2. An infinite model of τ.

$\mathfrak{M} \models w_{l-r}Rw$. But w_{l-r} satisfies p_2 or p_3, w satisfies p_0, and thus λ_2, λ_3 forbid this connection. Therefore there is no finite model of λ based on a frame from $\mathcal{K}_\Phi$. □

Before we show the algorithm that solves the satisfiability problem, we present a simple property of the finite global satisfiability problem, namely, that every satisfiable formula has a model which is strongly connected.

Lemma 3.4 *Let $\Phi \in \mathsf{UHF}$, φ be a modal formula and $\mathfrak{M}$ be a finite $\mathcal{K}_\Phi$-based model such that $\mathfrak{M} \models \varphi$. Then there is a $\mathcal{K}_\Phi$-based submodel $\mathfrak{N}$ of $\mathfrak{M}$ such that $\mathfrak{N} \models \varphi$ and the frame of $\mathfrak{N}$ is strongly connected.*

Proof. Consider a relation on the set of strongly connected components of $\mathfrak{M}$, defined in such a way that $\mathfrak{N} \leq \mathfrak{N}'$ iff there is a path from an element of $\mathfrak{N}'$ to an element of $\mathfrak{N}$, or if $\mathfrak{N} = \mathfrak{N}'$. It is not hard to see that $\leq$ is a partial order. Since $\mathfrak{M}$ is finite, there must be a component $\mathfrak{N}_{min}$ which is minimal with respect to $\leq$. As Φ is universal, $\mathfrak{N}_{min}$ satisfies Φ. Moreover, since each world from $\mathfrak{N}_{min}$ has all its successors in $\mathfrak{N}$ (there is no path to worlds in other connected components), $\mathfrak{N}_{min}$ satisfies φ. □

We say that a frame $\mathcal{M}$ is *k-periodic* if its universe can be divided into pairwise disjoint, non-empty sets of worlds $W_1, W_2, \ldots, W_k$ such that for each v, w from $\mathcal{M}$ there is an edge from v to w if and only if for some $i \leq k$, $v \in W_i$ and $w \in W_{(i \mod k)+1}$. Notice that a 1-periodic frame is a clique. For each $k \in \mathbb{N}$ we define the cycle $\mathcal{C}_k$ as $\mathcal{I}_k$ with one additional edge, namely $(\underline{k-1}, \underline{0})$. Clearly, each $\mathcal{C}_k$ is k-periodic.

We are going to prove decidability by showing that each satisfiable formula has a model that is k-periodic for some k. In order to do so, we introduce two technical lemmas.

Lemma 3.5 *Let $\Phi \in \mathsf{UHF}$.*

(a) If Φ has a k-periodic model $\mathcal{M}$, then $\mathcal{C}_k$ is a model of Φ.

(b) If $\mathcal{C}_k$ is a model of Φ, then any k-periodic frame is a model of Φ.

(c) If $\mathcal{L}_\mathbb{Z}$ is a model of Φ, then for all $c > |\Phi|$, $\mathcal{C}_c$ is a model of Φ.

(d) If for some $k > |\Phi|$ the frame $\mathcal{C}_k$ is a model of Φ, then $\mathcal{L}_\mathbb{Z}$ is a model of Φ.

Proof. For (a), observe that if a periodic model $\mathcal{M}$ that consists of sets W_1, $W_2, \ldots, W_k$ is a model of Φ, then $\mathcal{C}_k$ is isomorphic with an induced substructure of $\mathfrak{M}$ that contains one world from every W_i.

We say that a morphism $h : \mathcal{M} \to \mathcal{M}'$ is *complete* if for all v, v' we have $h(v)Rh(v')$ if and only if vRv'. Note that if there is a complete morphism $h : \mathcal{M} \to \mathcal{M}'$ and Φ does not hold in $\mathcal{M}$, then it does not hold in $\mathcal{M}'$.

For (b), assume that there is a periodic frame $\mathcal{M}$ that consists of sets W_1, W_2, $\ldots$, W_k and is not a model of Φ, but $\mathcal{C}_k$ is a model of Φ. We define a complete morphism $f : \mathcal{M} \to \mathcal{C}_k$ as $f(v) = \underline{i}$ for $v \in W_i$. Since Φ does not hold in $\mathcal{M}$ and f is a complete morphism, Φ does not hold in $\mathcal{C}_k$ — a contradiction.

We prove (c) as follows. Let $c > |\Phi|$. Assume that there is a clause Ψ satisfied in $\mathcal{L}_\mathbb{Z}$ but not in $\mathcal{C}_c$, and let $v_1, v_2, \ldots, v_n$ be worlds of $\mathcal{C}_c$ such that $\Psi(v_1, \ldots, v_n)$ is false. Let k be such that no world among $v_1, \ldots, v_n$ is equal $\underline{\mathrm{k}}$. Consider the function $f : \mathcal{C}_{c\restriction\{v_1,\ldots,v_n\}} \to \mathcal{L}_\mathbb{Z}$ defined as

$$f(\underline{s}) = \begin{cases} \underline{s} & \text{for } s > k \\ \underline{c+s} & \text{for } s < k \end{cases}$$

A quick check shows that the function f is a complete morphism. Since $\Psi(v_1, \ldots, v_n)$ does not hold in $\mathcal{C}_c$, it follows that $\Psi(f(v_1), \ldots, f(v_n))$ does not hold in $\mathcal{L}_\mathbb{Z}$. But $\mathcal{L}_\mathbb{Z} \models \Psi$, a contradiction.

For the proof of (d), let $k > |\Phi|$, $\Psi \Rightarrow \Psi'$ be satisfied in $\mathcal{C}_k$ but not in $\mathcal{L}_\mathbb{Z}$. Let $v_1 = \underline{s}, v_1 = \underline{t}, v_3 \ldots, v_n$ be worlds of $\mathcal{L}_\mathbb{Z}$ such that $\Psi(v_1, \ldots, v_n)$ is true, $\Psi'(v_1, \ldots, v_n)$ is not, and $|s-t|$ is minimal. Let $f(\underline{i}) = \underline{i \mod k}$ be a morphism from $\mathcal{L}_\mathbb{Z}$ onto $\mathcal{C}_k$. If $t - s \mod k \neq 1$, then $\Psi \Rightarrow \Psi'(f(v_1), \ldots, f(v_n))$ does not hold and we have a contradiction. Otherwise, $|s-t| \geq k-1$ so there is a world $\underline{l}$ such that l is between s and t and $\underline{l}$ is different from all of $\underline{s}, \underline{t}, v_3, \ldots, v_n$. But then, morphism $g : \mathcal{L}_{\mathbb{Z}\restriction\{v_1,\ldots,v_n\}} \to \mathcal{L}_\mathbb{Z}$ defined as $g(\underline{s}) = \underline{s}$ for $s < l$ and $g(\underline{s}) = \underline{s-1}$ leads to the contradiction with the minimality of $|s-t|$. ☐

Lemma 3.6 *Let Φ be an unbounded* UHF *formula that does not force long edges and such that in $\mathfrak{C}_\Phi(\mathcal{X})$ for some $i, j < 0$ we have $\overline{i}R\underline{j}$ or $\underline{i}R\overline{j}$. Then $j - i = 1$ and $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z}) = \mathcal{L}_\mathbb{Z}$.*

Proof. As $\mathcal{X}$ is symmetric, $\mathfrak{C}_\Phi(\mathcal{X}) \models \underline{i}R\overline{j}$ implies $\mathfrak{C}_\Phi(\mathcal{X}) \models \overline{i}R\underline{j}$. So we assume that $\mathfrak{C}_\Phi(\mathcal{X}) \models \underline{i}R\overline{j}$.

Let us consider a morphism f from $\mathcal{X}$ into $\mathcal{L}_\mathbb{Z}$ defined as

$$f(\underline{k}) = f(\overline{k}) = \underline{k}$$

If $|j - i| > 1$, then there is a long edge in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ and it contradicts the assumption that $\mathcal{X}$ does not force long edges.

If $j - i = -1$, then the morphism f implies that there is an edge $(\underline{j}, \underline{j-1})$ in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ and, since $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ is uniform, for all k there are edges $(\underline{k}, \underline{k-1})$ in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$. We define another morphism g to show that then $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ contains a long edge. Let g be a morphism from $\mathcal{X}$ into $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ defined as

$$g(w) = \begin{cases} \underline{|k|} & \text{if } w = \underline{k} \text{ for some } k \\ \underline{-|k|} & \text{if } w = \overline{k} \text{ for some } k \end{cases}$$

It is not hard to see that g is indeed a morphism and therefore that $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ contains a long edge $(\underline{|i|}, \underline{-|j|})$. An example is presented in Fig. 1.

If $j = i$, then the morphism f implies that there is a reflexive world in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$, and therefore all worlds are reflexive. Consider a morphism h from $\mathcal{X}$ into $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ defined as

$$h(w) = \begin{cases} \underline{1} & \text{if } w = \overline{k} \text{ for some } k \leq i \\ \underline{0} & \text{otherwise} \end{cases}$$

Since all worlds in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ are reflexive, h is indeed a morphism, so in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ there is edge $(\underline{1}, \underline{0})$ and, as in the previous case, all edges in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ are symmetric and therefore $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ contains a long edge.

For the proof of $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z}) = \mathcal{L}_\mathbb{Z}$, recall that if $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ contains a symmetric or reflexive edge, then it contains long edges. But Φ does not force long edges, and therefore $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z}) = \mathcal{L}_\mathbb{Z}$. □

In the proof of our next proposition we use the following simple fact, whose proof follows easily by an application of the Euclidean algorithm.

Fact 3.7 *Let X be a set of positive numbers. Then, there exists a finite subset X' of X such that $gcd(X) = gcd(X')$. Moreover, if X is closed under addition then for each $x > lcm(X')$, $gcd(X')$ divides x iff $x \in X$.*

For a given model $\mathfrak{M}$, we define a *characteristic cycle* of $\mathfrak{M}$ as a walk $v_0, v_1, \ldots, v_{l-1}$ that contains all worlds from $\mathfrak{M}$ and, moreover, in $\mathfrak{M}$ there is an edge from v_{l-1} to v_0. Note that for all strongly connected models containing at least two worlds such a characteristic cycle exists.

Proposition 3.8 *Let Φ be an unbounded* UHF *formula that does not force long edges, merges at a level $k < 0$ and forks at all levels. Then global $\mathcal{K}_\Phi$-FINSAT is in* PSpace.

Proof. Let φ be a modal formula and $\mathfrak{M}$ be a strongly connected model of φ from $\mathcal{K}_\Phi$. Such a model exists due to Lemma 3.4. Assume that $\mathfrak{M}$ contains at least two worlds and let $v_0, v_1, \ldots, v_{l-1}$ be a characteristic cycle of $\mathfrak{M}$. For better readability, below we omit " mod l" in subscripts of vs.

Our aim is to show that $\mathfrak{M}$ is s-periodic for some s.

Let $\mathcal{X}_\mathfrak{M} \subseteq \mathbb{N}$ be such that $k \in \mathcal{X}_\mathfrak{M}$ if and only if there is v_i such that $\mathfrak{M} \models v_i R v_{i+k+1}$. Lemma 3.2 implies that for all v_i and $k \in \mathcal{X}_\mathfrak{M}$, $\mathfrak{M} \models v_i R v_{i+k+1}$.

We show that $\mathcal{X}_\mathfrak{M}$ is additively closed. Assume that $x, y \in \mathcal{X}_\mathfrak{M}$. It means that $\mathfrak{M}$ contains edges (v_{x+y+1}, v_{x+y+2}) , (v_{x+1}, v_{x+y+2}) and (v_0, v_{x+1}). We define a morphism h from $\mathcal{X}$ to $\mathcal{M}$, the frame of $\mathfrak{M}$, as

$$h(w) = \begin{cases} v_s & \text{if } w = \underline{k-1+s} \text{ for all } s \leq 0 \\ v_{x+1} & \text{if } w = \underline{k} \\ v_{x+y+1+s} & \text{if } w = \underline{k+s} \text{ for all } s > 0 \\ v_{x+y+1+s} & \text{if } w = \overline{k+s} \text{ for all } s \in \mathbb{Z} \end{cases}$$

We see that $h(\underline{k-1}) = v_0$ and $h(\overline{k}) = v_{x+y+1}$, and since in $\mathfrak{M}$ there is an edge from $\underline{k-1}$ to $\overline{k}$, $x + y \in \mathcal{X}_{\mathfrak{M}}$.

Let $\mathcal{X}^l_{\mathfrak{M}} = \{i \mod l | i \in \mathcal{X}_{\mathfrak{M}}\}$. By Fact 3.7, $\mathcal{X}^l_{\mathfrak{M}}$ can be represented as $\{i \cdot gcd(\mathcal{X}_{\mathfrak{M}}) \mod l | i \in \mathbb{N}\}$. Define $W_i = \{v_{i+j\cdot gcd(\mathcal{X}_{\mathfrak{M}})} | j \in \mathbb{N}\}$. It follows that all elements of W_i have all successors in W_{i+1}, and therefore $\mathfrak{M}$ is $gcd(\mathcal{X}_{\mathfrak{M}})$-periodic.

Now we show how to compress sets W_i. For each i and each subformula ψ of φ, if there is a world in W_i that satisfies ψ, we mark one such world. Then we remove unmarked worlds. It is easy to see that the types of worlds remain the same.

We have proved that all models of φ are s-periodic and that their sets can be compressed to a size bounded by $|\varphi|$, but the value of s can be arbitrary large. Now we show that there is an NPSpace (=PSpace) procedure that checks, for a given modal formula φ, if φ has a Φ-based finite global periodic model.

Our NPSpace algorithm works as follows. First, it checks if there is a single world or a single clique (1-periodic set) with size bounded by $|\varphi|$, that satisfies both φ and Φ. If it is the case the algorithm returns "Yes". Otherwise, it guesses a set W_1 with size bounded by $|\varphi|$ and then, recursively, guesses the successive sets with size similarly bounded, checking if guessed worlds are consistent with their predecessor, and returns "no" otherwise. The algorithm stops after $\binom{2^{|\varphi|}}{|\varphi|} + 1$ steps and returns "yes".

If there is a model of φ, then the algorithm returns "yes". Indeed, we showed that φ has a single world model or an s-periodic model with size of sets bounded by $|\varphi|$, and the algorithm can simply guess this world or successively guess consecutive sets of this model.

If the algorithm returns "yes", then it visited two sets satisfying the same subformulas, so there is a sequence of sets $V_1, V_2, \ldots, V_k, V_1$ with $k \leq 2^{|\varphi|}$ such that each set contains all witnesses needed by its predecessors. We build an s-periodic model that contains sets $V_1, \ldots, V_k$ repeated $\lceil |\Phi|/k \rceil + 1$ times. Clearly, the obtained model satisfies φ. By Lemma 3.6, $\mathcal{L}_{\mathbb{Z}} = \mathfrak{C}_\Phi(\mathcal{L}_{\mathbb{Z}})$, and by Lemma 3.5 it is also a model of Φ. □

The corresponding lower bound can be shown by an encoding of a version of the corridor-tiling problem. A *tiling system* is a tuple $\mathcal{D} = (D, H_\mathcal{D}, V_\mathcal{D}, n)$, where D is a set of tiles, $H_\mathcal{D}, V_\mathcal{D} \subseteq D \times D$ are binary relations specifying admissible horizontal and vertical adjacencies, and n is a unary encoded natural number. For a given tiling system we ask if there exists a tiling of an infinite corridor of width n, respecting $H_\mathcal{D}$, $V_\mathcal{D}$ constraints. Formally, we ask if there exists a tiling $t : \{0, 1, \ldots, n-1\} \times \mathbb{N} \to D$, such that for all $0 \leq k < n$ and $l \in \mathbb{N}$ we have $(t(k, l), t(k, l+1)) \in V_\mathcal{D}$ and for all $0 \leq k < n-1$, $l \in \mathbb{N}$ we have $(t(k, l), t(k+1, l)) \in H_\mathcal{D}$. This problem is known to be PSpace-complete [24].

Proposition 3.9 *Let Φ be an unbounded* UHF *formula that does not force long edges, merges at a level $k < 0$ and forks at all levels. Then global $\mathcal{K}_\Phi$-FINSAT* *is* PSPACE-*hard.*

Proof. Let $\mathcal{D} = (D, H_\mathcal{D}, V_\mathcal{D}, n)$ be an instance of the corridor-tiling problem. We will construct a modal formula η which is globally, finitely $\mathcal{K}_\Phi$-satisfiable iff $\mathcal{D}$ has a solution. In our intended model a single world represents a whole row of a solution.

We employ propositional variables p_i^d for $i < n$ and $d \in \mathcal{D}$. The intended meaning of p_i^d is that the point in column i is tiled by d.

We put $\eta = \eta^l \wedge \eta^h \wedge \eta^v$, where η^l that guarantees that each point is tiled by exactly one element of $\mathcal{D}$, η^h ensures that the tilling respects $H_\mathcal{D}$, and η^v ensures that each world has a successor that describes the row which is consistent with the current one with respect to the relation $V_\mathcal{D}$.

$$\eta^l = \bigwedge_{i<n} (\bigvee_{d\in\mathcal{D}} p_i^d \wedge \bigwedge_{d,d'\in\mathcal{D}, d\neq d'} \neg(p_i^d \wedge p_i^{d'}))$$

$$\eta^h = \bigwedge_{i<n-1} \bigvee_{(d,d')\in H_\mathcal{D}} (p_i^d \wedge p_{i+1}^{d'})$$

$$\eta^v = \Diamond\top \wedge \bigwedge_{i<n} \bigvee_{(d,d')\in V_\mathcal{D}} (p_i^d \wedge \Box p_i^{d'})$$

Assume that $\langle \mathcal{D}, V_\mathcal{D}, H_\mathcal{D}, n\rangle$ has a solution that consists of rows $r_1, r_2, \ldots$. Then among first n^n+1 of them some rows r_i, r_j with $i < j$ are tiled identically. Let $l = c(j-i)$, for some $c > |\Phi|$. We encode the solution on $\mathcal{C}_l$ in such a way that $\underline{s}$ represents row $i + (s \mod l)$. Note that by Lemma 3.6 and Lemma 3.5(c) it follows that $\mathcal{C}_l$ belongs to $\mathcal{K}_\Phi$. Conversely, if η has a model $\mathfrak{M}$ then we can construct a solution by starting from an arbitrary world of $\mathfrak{M}$, translating it to the initial row of a solution in a natural way, and recursively building successive rows as translations of the worlds guaranteed by η^v. $\square$

Formulas that do not merge and fork at all levels. Now we prove that in the case of formulas Φ that do not force long edges, fork at all levels and do not merge at any level, modal logic has the finite model property with respect to $\mathcal{K}_\Phi$. In the proof, we start from an infinite tree–based model $\mathfrak{M}$, and construct a very large structure that locally looks like a part of $\mathfrak{M}$, but is finite. We need to do it carefully in order not to violate the first–order formula Φ.

Proposition 3.10 *Let Φ be an unbounded* UHF *formula that does not force long edges, forks at all levels and does not merge at any level $k < 0$. Then modal logic has the finite global model property with respect to $\mathcal{K}_\Phi$.*

Proof. Let $\mathfrak{M}^b$ be a tree-based model of φ and Φ, based on a tree $\mathcal{T}^b$, as guaranteed by Lemma 2.1. Let $n = |\varphi|$ and $N = |\Phi|$. If there is a world in $\mathfrak{M}^b$ without a proper successor, then the structure that contains only this world is a model of φ and Φ. Otherwise, all worlds are ∞-followed. We assume that every world has degree n – if a world has a smaller degree, then we can replicate any of its subtrees.

Let w be any $\mathfrak{g}(|\Phi|)$-inner world in $\mathcal{T}^b$, $\mathcal{T}$ be a subtree of $\mathcal{T}^b$ rooted at w, and $\mathfrak{M}$ be a substructure of $\mathfrak{M}^b$ that consists of the worlds from $\mathcal{T}$. Clearly, $\mathfrak{M}$ satisfies Φ and φ.

Let M be the universe of $\mathfrak{M}$. For each $w \in M$, we define a tree $\mathcal{S}'_w$ to be the subtree of $\mathcal{M}$ rooted in w, $\mathcal{S}_w$ to be the frame that contains first $2N$ levels of $\mathcal{S}'_w$, and $\mathfrak{S}_w$ to be the substructure of $\mathfrak{M}$ that contains the worlds from $\mathcal{S}_w$. Let $tp(\mathfrak{M})$ be a set of all types realized in $\mathfrak{M}$. For each type $t \in tp(\mathfrak{M})$, we pick one world w_t of this type and define $\mathfrak{S}_t = \mathfrak{S}_w$ and $\mathcal{S}_t = \mathcal{S}_w$

For each $\mathcal{S}_t$, we label leaves in $\mathcal{S}_t$ in a consecutive way, e.g. from left to right, such that leaves labeled with 1, 2, ..., n have the same parent and so on.

For each $s \in \{0,1\}$, $p \in \{1,\ldots,n\}$ and $t \in tp(\mathfrak{M})$, we define $\mathfrak{T}^s_{t,p}$ as a copy of $\mathfrak{S}_t$. We define the finite structure $\mathfrak{M}_s$ as a disjoint union of all possible $\mathfrak{T}^s_{t,p}$. We say that a world w is *at a level* k in $\mathfrak{T}^s_{t,p}$ if it is a copy of a world that is at a level k in $\mathcal{S}_t$ and that it is *at a level* k in $\mathfrak{M}_s$ if it is at a level k in some tree of $\mathfrak{M}_s$. We say that a world v is a *parent* of v' in $\mathfrak{M}_k$ if wRv, v is at a level k and v' is at a level $k+1$ for some k. For any two worlds v, v' that are in the same tree, we define $lca(v,v')$ as the lowest common ancestor of v and v' (w.r.t. the relation parent). We define $llca(v,v')$ as the level of $lca(v,v')$ if such world exists and $llca(v,v') = -1$ otherwise.

We define a structure $\mathfrak{M}'$ as a disjoint union of $\mathfrak{M}_0$ and $\mathfrak{M}_1$ with additional edges defined as follows. Consider tree $\mathfrak{T}^0_{t,p}$ and its leaf v labeled by p. Let w be a world in $\mathfrak{M}$ with the same type and $t_1,\ldots,t_k$ be types of successors of w in $\mathcal{T}$. For each $j \le k$ we add an edge from v to the root of $\mathfrak{T}^1_{t_j,p}$ and, if some connection between w and its successors is symmetric, we make this edge symmetric as well. We do the same for the leaves from $\mathfrak{M}_1$, but we connect them with the roots from $\mathfrak{M}_0$.

It is not hard to see that all worlds in $\mathfrak{M}'$ satisfy φ. We prove that $\mathfrak{M}'$ satisfies Φ. Assume to the contrary that this is not the case. Let $\Psi \Rightarrow \Psi'$ be a formula which is not satisfied in $\mathfrak{M}'$. Then there are worlds $v_1,\ldots,v_n$ such that $\Psi(v_1,\ldots,v_n)$ holds but $\Psi'(v_1,\ldots,v_n)$ does not.

We define a function $\nu_k : \mathfrak{M}' \to \{0,\ldots,4N-1\}$ as

$$\nu_k(v) = \begin{cases} s-k & \text{for each } v \text{ at a level } s \ge k \text{ in } M_0 \\ s+2N-k & \text{for each } v \text{ at a level } s \text{ in } M_1 \\ s+4N-k & \text{for each } v \text{ at a level } s < k \text{ in } M_0 \end{cases}$$

Let k be such that no world among $v_1,\ldots,v_n$ is at level k in $\mathfrak{M}_0$ and $\mathfrak{M}_1$. A function $f : \mathfrak{M}'\restriction_{\{v_1,\ldots,v_n\}} \to \mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ defined as

$$f(v) = \underline{\nu_k(v)}$$

is a morphism.

It is not possible that $\Psi' = \bot$, because then Φ would not be satisfied in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ and since Φ is unbounded, $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ is a model of Φ. Similarly, if $\Psi' = xRx$, then some world in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ would be reflexive and, since all worlds in $\mathfrak{M}$ are $\mathfrak{g}(|\Phi|)$-inner in $\mathfrak{M}^b$, $\Psi'(v_1,\ldots,v_n)$ would be satisfied.

The only remaining case is $\Psi' = xRy$. Let v_1 be at a level l_1 in $\mathfrak{M}_{s_1}$ and v_2 be at a level l_2 is $\mathfrak{M}_{s_2}$. There are two cases: either $s_1 = s_2$ and $|l_1 - l_2| \leq 1$, or $s_1 \neq s_2$ and one of v_1, v_2 is a root and the other one is a leaf. Otherwise, Φ would force long edges.

Assume that $s_1 < s_2$ and let k be such that no world among $v_1, \ldots, v_n$ is at a level k in $\mathfrak{M}_0$. Consider a morphism $g : \mathfrak{M}'{\restriction_{\{v_1,\ldots,v_n\}}} \to \mathfrak{M}'$ defined as

$$g(v) = \begin{cases} v' & \text{if } v \text{ is at a level } i \geq k \text{ in } \mathfrak{M}_0 \text{ and } v' \text{ is a parent of } v \\ v & \text{otherwise} \end{cases}$$

It implies that Φ requires also an edge from some world that is not a leaf to some root, and so by the morphism f we can show that Φ forces long edges. The case when $s_1 > s_2$ is symmetric.

Assume that $s_1 = s_2 = 0$. If $v_1 = v_2$, then, by the morphism f, all worlds of $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ are reflexive and Ψ' would be satisfied, as before. If v_2 is a parent of v_1, then, by the morphism f, all edges in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ are symmetric and Ψ' would be satisfied. So we can assume that v_1 and v_2 are not on the same path in $\mathfrak{M}_0$.

Assume that $l_1 \leq N$ and $l_2 \leq N$ and let $k > N$ be such that no world among $v_3, \ldots, v_N$ is at level k in $\mathfrak{M}_0$. We define a morphism $h_1 : \mathfrak{M}'{\restriction_{\{v_1,\ldots,v_n\}}} \to \mathcal{T}_\infty$ as follows.

$$h_1(v) = \begin{cases} \underline{0^{\nu_k(v)}} & \text{if } \nu_k(v) < 4N - k \\ \underline{0^{4N-k+llca(v,v_1)}1^{s-llca(v,v_1)}} & \text{if } v \text{ at level } s \text{ and } \nu_k(v) \geq 4N - k \end{cases}$$

Let $m = llca(v_1, v_2)$. Since v_1 and v_2 are not on the same path, $m < min(l_1, l_2)$. Since $h_1(v_1) = \underline{0^{4N-k+l_1}}$ and $h_1(v_2) = \underline{0^{4N-k+m}1^{l_2-m}}$ and h_1 is a morphism, it implies that Φ does not fork a the level $\underline{0^{4N-k+m}}$ — a contradiction.

Now consider the case when $l_1 \geq N$ and $l_2 \geq N$. Let $k < N$ be such that no world among $v_3, \ldots, v_N$ is at the level k in $\mathfrak{M}_0$.

If $llca(v_1, v_2) \leq k$, then Φ merges at some level. We prove it using the following morphism $h_2 : \mathfrak{M}'{\restriction_{\{v_1,\ldots,v_n\}}} \to \mathcal{X}$. Let $\mathfrak{T}^0_{t,p}$ be the tree that contains v_1.

$$h_2(v) = \begin{cases} \underline{s - 2N} & \text{if } v \text{ at a level } s \geq k \text{ in } \mathfrak{M}_0 \text{ and } llca(v_1, v) > k \\ \overline{\underline{s - 2N}} & \text{if } v \text{ at a level } s \geq k \text{ in } \mathfrak{M}_0 \text{ and } llca(v_1, v) \leq k \\ \underline{s} & \text{if } v \text{ at a level } s \text{ in } \mathfrak{M}_1 \\ \underline{2N + s} & \text{if } v \text{ at a level } s \text{ in } \mathfrak{M}_0 \end{cases}$$

It is readily checkable that h_2 is a morphism and it implies that Φ merges at some level.

Let $llca(v_1, v_2) > k$. We prove that Φ does not fork at some level. To this end, let k' be such that no world among $v_3, \ldots, v_N$ is at the level k' in $\mathfrak{M}_1$. We define $V_1 = V_{M0} \cup V_{M1}$ as follows. Set $v \in V_{M0}$ if and only if v is at a level $s > k$ in $\mathfrak{M}_0$ and $lcm(v_1, v) \in \{v_1, v\}$ (in other worlds, v is an ancestor or descendant of v_1 in $\mathfrak{M}_0$). Finally, for each leaf w from V_{M0} labeled by m and each $t \in tp(\mathfrak{M})$, V_{M1} contains all worlds from levels less than k' in $\mathfrak{T}^1_{t,m}$.

Let $t = llca(v_1, v_2) - k$, We define a morphism $h_3 : \mathfrak{M}'{\restriction_{\{v_1,\ldots,v_n\}}} \to \mathcal{T}_\infty$.

$$h_3(v) = \begin{cases} \underline{0^{\nu_k(v)}} & \text{if } v \in V_1 \text{ or } \nu_k(v) < t \\ \underline{0^t 1^{\nu_k(v)-t}} & \text{otherwise} \end{cases}$$

It is readily checkable that h_3 is a morphism and it implies that Φ does not fork at the level t.

The case when $s_1 = s_2 = 1$ is symmetric. □

Formulas that do not merge and do not forks at some level. In the case of formulas that do not force long edges and do not fork at some level, the finite model property follows from the fact that each satisfiable formula has a k-periodic model for some k.

Proposition 3.11 *Let Φ be an unbounded* UHF *formula that does not force long edges and does not fork at some level $k > 0$. Then modal logic has the finite global model property with respect to $\mathcal{K}_\Phi$.*

Proof. Let $\mathfrak{M}$ be defined as in the proof of Proposition 3.10. First, observe that $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z}) = \mathcal{L}_\mathbb{Z}$ and, since Φ is unbounded, $\mathcal{L}_\mathbb{Z}$ is a model of Φ. Let v be a world at level $\mathfrak{g}(|\Phi|)$ and let $\mathfrak{M}'$ be the model that consists of all descendants of v from levels greater than $2\mathfrak{g}(|\Phi|)$. By Lemma 2.7, all worlds in $\mathfrak{M}'$ at the same level are equivalent. Since the number of types is bounded, there exist two levels k, l in $\mathfrak{M}'$ such that $k - l > |\Phi| + 1$ and the sets of types realized at levels k and l are equal. We create model $\mathfrak{M}''$ by removing all worlds from levels greater than or equal to k, and connecting all worlds from level $k-1$ to worlds from level l. Finally, we define $\mathfrak{M}'''$ by taking for each level one world of each type realized at this level. A quick check shows that models $\mathfrak{M}'$, $\mathfrak{M}''$, and $\mathfrak{M}'''$ satisfy φ and that $\mathfrak{M}'''$ is finite.

Now we justify that $\mathfrak{M}'''$ is a model of Φ. Since $\mathcal{L}_\mathbb{Z}$ is a model of Φ, Lemma 3.5 shows that C_{k-l} is a model of Φ, and the same lemma shows that therefore any $k{-}l$–periodic model is a model of Φ. Model $\mathfrak{M}'''$ is obviously $k{-}l$-periodic.□

4 Formulas that force long edges

As mentioned earlier, for formulas that satisfy S3, the polynomial model property follows from [16]. The rest of this section is devoted to formulas $\Phi \in$ UHF that satisfy S2.

First, observe that in this case modal logic may lack the finite model property (local and global) with respect to $\mathcal{K}_\Phi$. Consider, for example, $(xRz_1 \wedge z_1Ry \Rightarrow xRy) \wedge (xRx \Rightarrow \bot)$ and a modal formula $\Diamond\top \wedge \Box\Diamond\top$. A quick check shows that all models of these formulas are infinite (in local and global cases). On the other hand, modal logic has the finite model property with respect to the class defined by $xRx \wedge (xRz_1 \wedge z_1Ry \Rightarrow xRy)$.

To show decidability we prove that if a formula φ has a finite model (in local or global case), then it has a model of size bounded by $|\varphi|^{O(|\varphi|)}$. Clearly, it leads to a NExpTime algorithm that simply guesses such a model and verifies it.

Consider a modal formula φ and its $\mathcal{K}_\Phi$-based model $\mathfrak{M}$ with universe M. We say that a world w is *redundant* for φ and $\mathfrak{M}$ if $\mathfrak{M}_{\restriction M \setminus \{w\}}$ is a model of φ. We prove the following lemma by showing that a model that is large enough has to contain a redundant world.

Lemma 4.1 *Let Φ be an unbounded* UHF *formula that forces long edges. If φ has a finite $\mathcal{K}_\Phi$-based model, then it has a $\mathcal{K}_\Phi$-based model of size bounded by $|\varphi|^{O(|\varphi|)}$.*

Proof. Let Φ be an unbounded UHF formula that satisfies S2 for some l and $a_1, \ldots, a_l$, and φ be a modal formula with a $\mathcal{K}_\Phi$-based model $\mathfrak{M}$.

Let $c = a_1$. Observe that for all $i \in \mathbb{Z}$ and $k \geq 0$ we have $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z}) \models \underline{i} R \underline{i + kc + 1}$.

We start from bounding the number of worlds that are not $\mathfrak{g}(|\Phi|)$-preceded. We use the standard selection technique [2] — we start from an arbitrary world that satisfies φ, and then recursively for each world added in the previous stage we pick at most $|\varphi|$ witnesses. Let $\mathfrak{M}'$ be a model obtained this way. We define *the royal part* of $\mathfrak{M}'$ as the set of worlds that contain all worlds that are not $\mathfrak{g}(|\Phi|)$-preceded and *the court* as the set of $\mathfrak{g}(|\Phi|)$-preceded worlds that were added as witnesses for some worlds from the royal part. Clearly, the total size of the royal part and the court can be bounded by $|\varphi|^{\mathfrak{g}(|\Phi|)+1}$.

Let w be a $\mathfrak{g}(|\Phi|)$-inner world not from the court such that for each subformula $\Diamond\psi$ of φ such that ψ is satisfied in w there exists a $\mathfrak{g}(|\Phi|)$-inner world $w_\psi \neq w$ that satisfies ψ and that there is a path from w to w_ψ with the length cj for some j. We show that w is redundant.

Consider any predecessor w' of w. If w' is not $\mathfrak{g}(|\Phi|)$-preceded, then it has all the required witnesses in the court and the royal part. Otherwise, let ψ be a subformula of φ such that w satisfies ψ. We show that there is an edge from w' to w_ψ. To this end, consider a path $v_1, v_2, \ldots, v_{\mathfrak{g}(|\Phi|)}$, w', w, $v'_1, v'_2, \ldots, v'_{cj}$, $w_\psi, v''_1, v''_2, \ldots, v''_{\mathfrak{g}(|\Phi|)}$. Such a path exists since w' is $\mathfrak{g}(|\Phi|)$-preceded and w_ψ is $\mathfrak{g}(|\Phi|)$-inner, and there is a straightforward morphism from $\mathcal{I}_{2\mathfrak{g}(|\Phi|)+2+cj}$ into this path. So it is enough to show that there is an edge from $\underline{\mathfrak{g}(|\Phi|)+1}$ to $\underline{\mathfrak{g}(|\Phi|)+1+cj+1}$ in $\mathfrak{C}(\mathcal{I}_{2\mathfrak{g}(|\Phi|)+2+j})$. By earlier observations, $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ contains an edge from $\underline{\mathfrak{g}(|\Phi|)+1}$ to $\underline{\mathfrak{g}(|\Phi|)+1+cj+1}$, and Lemma 2.6 implies that there is an edge from $\underline{\mathfrak{g}(|\Phi|)+1}$ to $\underline{\mathfrak{g}(|\Phi|)+1+cj+1}$ in $\mathfrak{C}(\mathcal{I}_{2\mathfrak{g}(|\Phi|)+2+cj})$.

By iterating the above argument we can remove all $\mathfrak{g}(|\Phi|)$—inner worlds except for at most $|\varphi|^{c \cdot |\varphi|}$ worlds. Finally, we again use the selection technique to bound the number of worlds that are not $\mathfrak{g}(|\Phi|)$-followed by $|\varphi|^{c \cdot |\varphi|} \cdot |\varphi|^{\mathfrak{g}(|\Phi|)}$. Since Φ is not a part of an instance, we reduced the number of worlds to $|\varphi|^{O(|\varphi|)}$. □

The above lemma leads to the following result.

Proposition 4.2 *If Φ is an unbounded* UHF *formula that forces long edges, then $\mathcal{K}_\Phi$-FINSAT and global $\mathcal{K}_\Phi$-FINSAT are in* NExpTime.

Establishing better complexity bounds in the case of formulas satisfying S2 is left as an open problem.

References

[1] Bárány, V. and M. Bojanczyk, *Finite satisfiability for guarded fixpoint logic*, Information Processing Letters **112** (2012), pp. 371–375.

[2] Blackburn, P., M. de Rijke and Y. Venema, "Modal Logic," Cambridge Tracts in Theoretical Comp. Sc. **53**, Cambridge University Press, Cambridge, 2001.

[3] Gabbay, D. M. and V. B. Shehtman, *Products of modal logics, part 1*, Logic Journal of the IGPL **6** (1998), pp. 6–1.

[4] Grädel, E., *On the restraining power of guards*, Journal of Symbolic Logic **64** (1999), pp. 1719–1742.

[5] Grädel, E., M. Otto and E. Rosen, *Two-variable logic with counting is decidable*, in: *Twelfth Annual IEEE Symposium on Logic in Computer Science*, 1997, pp. 306–317.

[6] Grädel, E. and I. Walukiewicz, *Guarded fixed point logic*, in: *Fourteenth Annual IEEE Symposium on Logic in Computer Science*, 1999, pp. 45–54.

[7] Hemaspaandra, E., *The price of universality*, Notre Dame Journal of Formal Logic **37** (1996), pp. 174–203.

[8] Hemaspaandra, E. and H. Schnoor, *On the complexity of elementary modal logics*, in: *STACS*, LIPIcs **1** (2008), pp. 349–360.

[9] Hemaspaandra, E. and H. Schnoor, *A universally defined undecidable unimodal logic*, in: *MFCS*, Lecture Notes in Computer Science **6907** (2011), pp. 364–375.

[10] Kieroński, E., J. Michaliszyn and J. Otop, *Modal logics definable by universal three-variable formulas*, in: *FSTTCS*, LIPIcs **13** (2011), pp. 264–275.

[11] Kieroński, E., J. Michaliszyn, I. Pratt-Hartmann and L. Tendera, *Extending two-variable first-order logic with equivalence closure*, in: *LICS '12: Proceedings of the 29th IEEE symposium on Logic in Computer Science*, 2012.

[12] Kieroński, E. and M. Otto, *Small substructures and decidability issues for first-order logic with two variables.*, in: *LICS*, 2005, pp. 448–457.

[13] Kieroński, E. and L. Tendera, *On finite satisfiability of the guarded fragment with equivalence or transitive guards*, LPAR **4790** (2007), pp. 318–332.

[14] Kieroński, E. and L. Tendera, *On finite satisfiability of two-variable first-order logic with equivalence relations*, LICS (2009), pp. 123–132.

[15] Michaliszyn, J., *Decidability of the guarded fragment with the transitive closure*, in: *ICALP (2)*, Lecture Notes in Computer Science **5556** (2009), pp. 261–272.

[16] Michaliszyn, J. and J. Otop, *Decidable elementary modal logics*, in: *LICS '12: Proceedings of the 29th IEEE symposium on Logic in Computer Science*, 2012.

[17] Mortimer, M., *On languages with two variables*, Mathematical Logic Quarterly **21** (1975), pp. 135–140.

[18] Otto, M., *Two variable first-order logic over ordered domains*, Journal of Symbolic Logic **66** (1998), pp. 685–702.

[19] Pacholski, L., W. Szwast and L. Tendera, *Complexity of two-variable logic with counting*, in: *Twelfth Annual IEEE Symposium on Logic in Computer Science*, 1997, pp. 318–327.

[20] Pratt-Hartmann, I., *Complexity of the two-variable fragment with (binary-coded) counting quantifiers*, Journal of Logic, Language and Information (2005).

[21] Sahlqvist, H., *Completeness and correspondence in the first and second order semantics for modal logic*, Proceedings of the Third Scandinavian Logic Symposium (1973).

[22] Szwast, W. and L. Tendera, *FO^2 with one transitive relation is decidable*, unpublished.

[23] Szwast, W. and L. Tendera, *The guarded fragment with transitive guards*, Annals of Pure and Applied Logic **128** (2004), pp. 227–276.

[24] van Emde Boas, P., *The convenience of tilings*, in: *Complexity, Logic, and Recursion Theory* (1997), pp. 331–363.

[25] Vardi, M. Y., *Why is modal logic so robustly decidable?*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, **31** (1997), pp. 149–184.