Decidable Elementary Modal Logics

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***Abstract*—In this paper, the modal logic over classes of structures definable by universal first-order Horn formulas is studied. We show that the satisfiability problems for those logics are decidable, confirming the conjecture from [1]. We provide a full classification of logics defined by universal first-order Horn formulas, with respect to the complexity of satisfiability of modal formulas over the classes of frames they define. It appears, that except for the trivial case of inconsistent formulas for which the problem is in P, local satisfiability is either NP-complete or PSPACE-complete, and global satisfiability is NP-complete, PSPACE-complete, or EXPTIME-complete. While our results hold even if we allow to use equality, we show that inequality leads to undecidability.**

I. INTRODUCTION

Modal logic for almost a hundred year has been an important topic in many academic disciplines, including philosophy, mathematics, linguistics and computer science. Currently it seems to be most intensively investigated by computer scientists. Among numerous branches in which modal logic, sometimes in disguise, finds applications, are hardware and software verification, cryptography and knowledge representation.

Modal logic was introduced by philosophers to study modes of truth. The idea was to extend propositional logic by some new constructions, of which two most important were $\Diamond\varphi$ and $\Box\varphi$, originally read as φ *is possible* and φ *is necessary*, respectively. A typical question was, given a set of axioms $\mathcal{A}$, corresponding usually to some intuitively acceptable aspects of truth, what is the logic defined by $\mathcal{A}$, i.e. which formulas are provable from $\mathcal{A}$ in a Hilbert-style system.

One of the most important steps in the history of modal logic was the invention of a formal semantics based on the notion of the so-called Kripke structures. Basically, a Kripke structure is a directed graph, called a *frame*, together with a valuation of propositional variables. Vertices of this graph are called *worlds*. For each world truth values of all propositional variables can be defined independently. In this semantics, $\Diamond\varphi$ means *the current world is connected to some world in which* φ *is true*; and $\Box\varphi$, equivalent to $\neg\Diamond\neg\varphi$, means φ *is true in all worlds to which the current world is connected.*

It appeared that there is a beautiful connection between syntactic and semantic approaches to modal logic [2]: logics defined by axioms can be equivalently defined by restricting classes of frames. E.g., the axiom $\Diamond\Diamond P \to \Diamond P$ (*if it is possible that* P *is possible, then* P *is possible*), is valid precisely in the class of transitive frames; the axiom $P \to \Diamond P$ (*if* P *is true, then* P *is possible*) – in the class of reflexive frames, $P \to \Box\Diamond P$ (*if* P *is true, then it is necessary that* P *is possible*) – in the class of symmetric frames, and the axiom $\Diamond P \to \Box\Diamond P$ (*if* P *is possible, then it is necessary that* P *is possible*) – in the class of Euclidean frames.

Thus we may think that every modal formula φ defines a class of frames, namely the class of those frames in which φ is valid. A formula φ is valid in a frame K if for any possible truth-assignment of propositional variables to the worlds of K, φ is true at every world. To express this definition we require second-order logic, since it involves quantification over sets of elements: for each variable P and a subset V of the set of worlds we have to consider the case in which P is true exactly in the worlds from V. Note however, that many important classes of frames, in particular all the classes we mentioned above, can be defined by simple first-order formulas. For a given first-order sentence Φ over the signature consisting of a single binary symbol R we define $\mathcal{K}_\Phi$ to be the set of those frames which satisfy Φ.

It is not hard to see that some modal logics defined by a first-order formula are undecidable. A stronger result was presented in [3]—it was shown that there exists a universal first-order formula with the equality such that the global satisfiability problem over the frames that satisfy this formula is undecidable. In [4], this result was improved — it was shown that the equality is not necessary. The proof from [4] works also for local satisfiability. Finally, in [5] it was shown that even a very simple formula with three variables without the equality may lead to undecidability.

Decidability for various classes of frames can be shown by employing the so-called standard translation of model logic to first-order logic. Indeed, the satisfiability of a modal formula φ in $\mathcal{K}_\Phi$ is equivalent to satisfiability of $st(\varphi) \wedge \Phi$, where $st(\varphi)$ is the standard translation of φ. In this way, we can show that even multimodal logic is decidable in any class defined by two-variable logic [6], even extended with linear order [7] or equivalence closures of two distinguished binary relations [8]. The same holds for formulas of the guarded fragment [9], even if we allow for some restricted application of fixed-points [10] and transitive closures [11]. The complexity bounds obtained this way, however, are high — usually between EXPTIME and 2NEXPTIME.

The classes of frames we mentioned earlier, i.e. transitive, reflexive, symmetric and Euclidean are decidable. They can be defined by first-order sentences even if we further restrict the language to universal Horn formulas, UHF. Universal Horn

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formulas were considered in [1], where a dichotomy result was proved, that the satisfiability problem for modal logic over the class of structures defined by an UHF formula (with an arbitrary number of variables) is either in NP or PSPACE-hard. The authors of [1] conjecture that the problem is decidable in PSPACE for all universal Horn formulas. We confirm this conjecture.

Theorem 1: Let Φ be a UHF sentence. Then the local and the global satisfiability problems for modal logic over $\mathcal{K}_\Phi$ are in PSPACE and EXPTIME, resp.

This theorem extends the decidability results for the classes we mentioned earlier in this introduction, in particular for modal logics T, B, K4, S4, S5. It also works for some interesting classes of frames, for which, up to our knowledge, decidability has not been established so far. An example is the class defined by $\forall xyzv(xRy \wedge yRz \wedge zRv \rightarrow xRv)$.

To complete the discussion, we study the case of Horn formulas with the equality ($\mathsf{UHF}^=$) and inequality ($\mathsf{UHF}^{\neq}$). We prove the following proposition.

Theorem 2: Let Φ be a $\mathsf{UHF}^=$ sentence. Then the local and the global satisfiability problems for modal logic over $\mathcal{K}_\Phi$ are in PSPACE and EXPTIME, resp.

For the complete picture, we also study formulas with inequality. We prove the following theorem.

Theorem 3: There exist formulas $\Gamma \in \mathsf{UHF}^{\neq}$, $\Gamma' \in \mathsf{UHF}^{\neq}$ such that the global satisfiability problem for modal logic over $\mathcal{K}_\Gamma$ and the local satisfiability problem for modal logic over $\mathcal{K}_{\Gamma'}$ are undecidable.

The paper is organized as follows. Section II contains precise formulation the decision problems that we consider and other basic definitions. Section III is devoted to our crucial *tree–based model property*. In Section IV we discuss some properties of tree-based models. Section V contains proof of Theorem 1 followed by a discussion on the complexity. Finally, Sections VI and VII contain proofs of Theorems 2 and 3, resp.

Related work. In [1], membership in PSPACE is shown for a rich subclass of UHF, including in particular all formulas which imply reflexivity. However, the problem remained open for formulas involving variants of transitivity. In [5] it is proved that the subclass of UHF that consists of the formulas with three variables is decidable. The beginning of our proof is similar to the proof in [5] — we start with a *closure* of a result of some unraveling. Then, in [5] they study, in some sense, all possible properties that can be defined — it is possible, because there is a finite number of non-equivalent formulas with three variables. With the unbounded number of variables it is no longer true, and therefore a more general approach is needed. Moreover, the paper [5] does not consider equality and inequality.

II. Preliminaries

As we work with both first-order logic and modal logic we help the reader to distinguish them in our notation: we denote first-order formulas with Greek capital letters, and modal formulas with Greek small letters. We assume that the reader is familiar with first-order logic and propositional logic.

Modal logic extends propositional logic with the operator $\Diamond$ and its dual $\Box$. Formulas of modal logic are interpreted in Kripke structures, which are triples of the form $\langle W, R, \pi \rangle$, where W is a set of worlds, $\langle W, R \rangle$ is a directed graph called a *frame*, and π is a function that assigns to each world a set of propositional variables which are true at this world. We say that a structure $\langle W, R, \pi \rangle$ is *based* on the frame $\langle W, R \rangle$. For a given class of frames $\mathcal{K}$, we say that a structure is $\mathcal{K}$-based if it is based on some frame from $\mathcal{K}$. We will use calligraphic letters $\mathcal{M}, \mathcal{N}$ to denote frames and Fraktur letters $\mathfrak{M}, \mathfrak{N}$ to denote structures.

For a frame $\langle W, R \rangle$ and a subset $W' \subseteq W$, we define $R_{\restriction W'} = R \cap (W' \times W')$. Similarly, for a labeling function π, we define $\pi_{\restriction W'}$ to be such that $\pi_{\restriction W'}(w) = \pi(w)$ for all $w \in W'$ and $\pi_{\restriction X}$ to be such that $\pi_{\restriction X}(w) = \pi(w) \cap X$. We define a restriction of a frame $\langle W, R \rangle_{\restriction W'}$ for $W' \subseteq W$ as $\langle W', R_{\restriction W'} \rangle$.

The semantics of modal logic is defined recursively. A modal formula φ is (locally) *satisfied* in a world w of a model $\mathfrak{M} = \langle W, R, \pi \rangle$, denoted as $\mathfrak{M}, w \models \varphi$ if

- (i) $\varphi = p$ where p is a variable and $\varphi \in \pi(w)$,
- (ii) $\varphi = \neg p$ where p is a variable and $\varphi \notin \pi(w)$,
- (iii) $\varphi = \varphi_1 \vee \varphi_2$ and $\mathfrak{M}, w \models \varphi_1$ or $\mathfrak{M}, w \models \varphi_2$,
- (iv) $\varphi = \varphi_1 \wedge \varphi_2$ and $\mathfrak{M}, w \models \varphi_1$ and $\mathfrak{M}, w \models \varphi_2$,
- (v) $\varphi = \Diamond \varphi'$ and there exists a world $v \in W$ such that $(w, v) \in R$ and $\mathfrak{M}, v \models \varphi'$,
- (vi) $\varphi = \Box \varphi'$ and for all worlds $v \in W$ such that $(w, v) \in R$ we have $\mathfrak{M}, v \models \varphi'$.

Note that in this paper all formulas are in the negation normal form. By $|\varphi|$ denote the length of φ. We say that a formula φ is *globally* satisfied in $\mathfrak{M}$, denoted as $\mathfrak{M} \models \varphi$, if for all worlds w of $\mathfrak{M}$, we have $\mathfrak{M}, w \models \varphi$.

For a given class of frames $\mathcal{K}$, we say that a formula φ is *locally* (resp. *globally*) $\mathcal{K}$-*satisfiable* if there exists a frame $K \in \mathcal{K}$, a structure $\mathfrak{M}$ based on K, and a world $w \in W$ such that $\mathfrak{M}, w \models \varphi$ (resp. $\mathfrak{M} \models \varphi$).

For a given formula φ, a Kripke structure $\mathfrak{M}$, and a world $w \in W$ we define the *type* of w (with respect to φ) in $\mathfrak{M}$ as $tp^{\varphi}_{\mathfrak{M}}(w) = \{\psi : \mathfrak{M}, w \models \psi$ and ψ is subformula of $\varphi\}$. We write $tp_{\mathfrak{M}}(w)$ if the formula is clear from context. Note that $|tp^{\varphi}_{\mathfrak{M}}(w)| \leq |\varphi|$.

We say that a world w is k-*followed* (k-*proceeded*) in the frame $\mathcal{M}$, if there exists a directed path $(w, u_1, u_2, \ldots, u_k)$ (resp. $(u_1, u_2, \ldots, u_k, w)$) in $\mathcal{M}$. We say that a world w is k-inner in $\mathcal{M}$ if it is k-proceeded and k-followed.

The set of *universal Horn formulas*, UHF, is defined as the set of those Φ over the language $\{R\}$ which are of the form $\forall \vec{x}.\Phi_1 \wedge \Phi_2 \wedge ... \wedge \Phi_i$, where each Φ_i is a Horn clause. A Horn clause is a disjunction of literals of which at most one is positive. We usually present Horn clauses as implications. For example, the formula $\forall xyz.(xRy \wedge yRz \Rightarrow xRz) \wedge (xRx \Rightarrow \bot)$ defines the set of transitive and irreflexive frames. We often skip the quantifiers and represent such formulas as a set of clauses, e.g.: $\{xRy \wedge yRz \Rightarrow xRz, xRx \Rightarrow \bot\}$. We assume without loss of generality that each Horn clause consists of variables x, y and $z_1, z_2, \ldots$, and is of the form $\Psi \Rightarrow \bot$, $\Psi \Rightarrow xRx$, or $\Psi \Rightarrow xRy$. We define $\Phi(v_x, v_y, v_1, \ldots, v_k)$ as the instantiation of Φ with $x = v_x$, $y = v_y$, $z_1 = v_1$, $z_2 = v_2$, and so on, e.g. $(xRz_1 \wedge z_1Rz_2 \wedge z_2Ry \Rightarrow xRy)(a, b, c, d) = aRc \wedge cRd \wedge dRb \Rightarrow aRb$. We denote by Φ^p the set of the clauses from Φ containing a positive literal, i.e. all clauses of Φ except those of the form $\Psi \Rightarrow \bot$.

We define the *local* (resp. *global*) *satisfiability problem* $\mathcal{K}$-SAT (resp. global $\mathcal{K}$-SAT) as follows. For a given modal formula, is this formula locally (resp. globally) $\mathcal{K}$-satisfiable? For a given $\Phi \in$ UHF, we define $\mathcal{K}_\Phi$ as the class of frames satisfying Φ. We will be interested in local and global $\mathcal{K}_\Phi$-SAT problems.

When considering problems $\mathcal{K}_\Phi$-SAT and global $\mathcal{K}_\Phi$-SAT, formula Φ is fixed and does not depend on the input. However, the complexity depends on this formula. To hide unnecessary details, we often use a function $\mathfrak{g}$ to bound the size of models or complexity in the size of Φ. Please keep in mind that once Φ is fixed, $\mathfrak{g}(|\Phi|)$ can be treated as a constant, and therefore the precise value of $\mathfrak{g}$ is not important (it will follow from the proofs).

III. Minimal tree-based models

In this section, we show that for every UHF formula Φ and every modal formula φ, if φ is $\mathcal{K}_\Phi$-satisfiable then it has a "nice" model. We start from an arbitrary $\mathfrak{M} \models \varphi$ satisfying Φ and unravel it (using standard unraveling technique, as in [2] and [12]) into a model $\mathfrak{M}_0$ whose frame is a tree with the degree of its nodes bounded by $|\varphi|$. Clearly the frame of $\mathfrak{M}_0$ is not necessarily a member of $\mathcal{K}_\Phi$. In the next step we add to $\mathfrak{M}_0$ the edges implied by the Horn clauses of Φ. This is performed in countably many stages, until the least fixed point is reached. We observe that the resulting structure, $\mathfrak{M}_\infty$, is still a model of φ, and its frame belongs to $\mathcal{K}_\Phi$.

Formally, we say that an edge (w, w') is a *consequence* of Φ in $\mathcal{M} = \langle W, R\rangle$, if for some worlds $v_1, \ldots, v_k \in W$ and $\Psi_1 \Rightarrow \Psi_2 \in \Phi$ we have $\mathcal{M} \models \Psi_1(w, w', v_1, \ldots, v_k)$, and $\Psi_2(w, w', v_1, \ldots, v_k) = wRw'$. We denote the set of all consequences of Φ in $\mathcal{M}$ by $\mathrm{C}^\Phi_\rightsquigarrow(\mathcal{M})$. We define the *consequence operator* as follows.

$\textsc{Cons}_{\Phi,W}(R) = R \cup \mathrm{C}^\Phi_\rightsquigarrow(\langle W, R\rangle)$

Now, the *closure operator* can be defined as the least fixed-point of $Cons$:

$\textsc{Closure}_{\Phi,W}(R) = \bigcup_{i>0} \textsc{Cons}^i_{\Phi,W}(R)$

Example 4: Consider the tree $\langle W, R\rangle$ presented in Fig. 1 and $\Phi = \{xRz \wedge zRy \Rightarrow yRy, xRx \wedge xRy \wedge xRz \Rightarrow yRz\}$. Bottom part of Fig. 1 contains the closure of this tree. Reflexive edges belong to $\textsc{Cons}_{\Phi,W}(R)$, dashed edges belong to $\textsc{Cons}^2_{\Phi,W}(R)$, and dotted edges belong to $\textsc{Cons}^3_{\Phi,W}(R)$. Quick check shows that $\textsc{Cons}^3_{\Phi,W}(R) = \textsc{Cons}^4_{\Phi,W}(R)$ and therefore $\textsc{Cons}^3_{\Phi,W}(R)$ is equal to $\textsc{Closure}_{\Phi,W}(R)$.

For a tree $\mathcal{T} = \langle W, R\rangle$, we now define the $\mathcal{T}$*-based model of* Φ as $\mathfrak{C}_\Phi(\mathcal{T}) = \langle W, \textsc{Closure}_{\Phi,W}(R)\rangle$. Note that $\mathfrak{C}_\Phi(\mathcal{T})$ is the smallest (w.r.t. inclusion of the set of edges) model of Φ^p

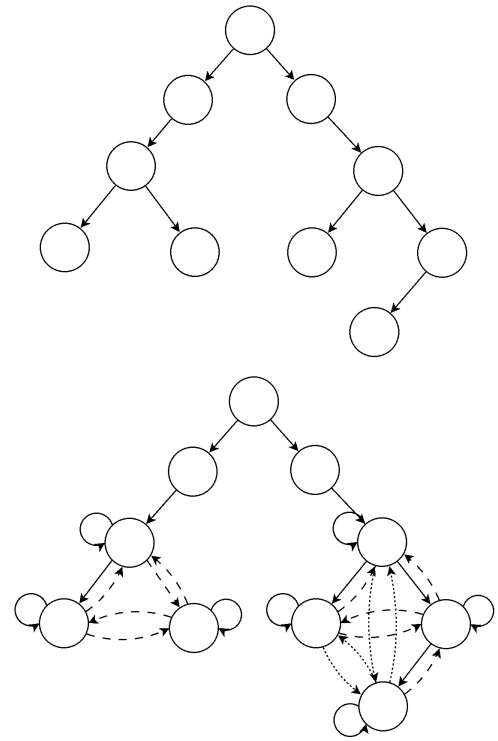

Figure 1. A closure for $\Phi = \{xRz \wedge zRy \Rightarrow yRy, xRx \wedge xRy \wedge xRz \Rightarrow yRz\}$.

containing all edges from R. Of course, not all models can be obtained in this way. The following lemma shows, however, that we can restrict our attention to models that are $\mathcal{T}$*-based* for some tree $\mathcal{T}$ with bounded degree.

Lemma 5: Let φ be a modal formula and let $\Phi \in$ UHF. If φ is $\mathcal{K}_\Phi$-satisfiable, then there exists a tree $\mathcal{T}$ with the degree bounded by $|\varphi|$ and a labeling $\pi_\mathcal{T}$, such that

(i) $\langle \mathcal{T}, \pi_\mathcal{T}\rangle$ is a model of φ;
(ii) $\langle \mathfrak{C}_\Phi(\mathcal{T}), \pi_\mathcal{T}\rangle$ is a model of φ that satisfies Φ.

The result holds for the local satisfiability and for the global satisfiability.

IV. Model properties

We study several properties of models. The following frames will be useful.

Definition 6: We define the linear structure $\mathcal{L}_\mathbb{Z}$ as $\langle\{\underline{i} : i \in \mathbb{Z}\}, \{(\underline{i}, \underline{i+1}) | i \in \mathbb{Z}\}\rangle$, and the infinite binary tree $\mathcal{T}_\infty$ as $\langle\{\underline{s} | s \in \{0,1\}^*\}, \{(\underline{s}, \underline{si}) | s \in \{0,1\}^* \wedge i \in \{0,1\}\}\rangle$.

The structures $\mathcal{L}_\mathbb{Z}$ and $\mathcal{T}_\infty$ play a crucial role in our proofs. We often reason in the following way. If for some $\mathcal{T}$ a property P is satisfied in a world of $\mathfrak{C}_\Phi(\mathcal{T})$, then we show that it is also satisfied in some world of $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ or $\mathfrak{C}_\Phi(\mathcal{T}_\infty)$. Thanks to the uniformity of those structures we show that the property P is satisfied in all $\mathfrak{g}(|\Phi|)$-proceeded worlds. Then we show that P has to be satisfied in all $\mathfrak{g}(|\Phi|)$-inner worlds of $\mathfrak{C}_\Phi(\mathcal{T})$.

Now we define our most important tool. We say that a function f from $\mathcal{M}_1$ into $\mathcal{M}_2$ is a *morphism* iff for all worlds w, w' if $\mathcal{M}_1 \models wRw'$, then $\mathcal{M}_2 \models f(w)Rf(w')$.

Observation 7: Let $\mathcal{M}_1, \mathcal{M}_2$ be frames, let $\Phi \in \mathsf{UHF}$ and let f be a function from $\mathcal{M}_1$ into $\mathcal{M}_2$. If f is a morphism from $\mathcal{M}_1$ into $\mathcal{M}_2$, then f is a morphism from $\mathfrak{C}_\Phi(\mathcal{M}_1)$ into $\mathfrak{C}_\Phi(\mathcal{M}_2)$.

We use morphisms to transfer properties between $\mathfrak{C}_\Phi(\mathcal{T})$ and $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ or $\mathfrak{C}_\Phi(\mathcal{T}_\infty)$. One morphism, we often use, is $h_\mathcal{T} : \mathcal{T} \to \mathcal{L}_\mathbb{Z}$ defined such that for each v at the ith level of $\mathcal{T}$, $h_\mathcal{T}(v) = \underline{i}$. Now we define an important property that tells us whether an UHF formula enforces edges between different branches of a tree.

Definition 8: We say that a formula $\Phi \in \mathsf{UHF}$ *forks at the level* i if for all $\underline{s} \in \mathcal{T}_\infty$ with $|s| = i$ and $t, t' \in \{0,1\}^*$ there is no edge between $\underline{s0t}$ and $\underline{s1t'}$ in $\mathfrak{C}_\Phi(\mathcal{T}_\infty)$. We say that $\Phi \in \mathsf{UHF}$ has the *tree-compatible model property* (TCMP) if for each i, Φ forks at the level i.

It is not hard to see that if Φ has the tree-compatible model property, then in all tree-based models of Φ there are no edges among the worlds from disjoint subtrees. Indeed, if there is an edge between two different subtrees $\mathcal{S}_1, \mathcal{S}_2$ of a model $\mathcal{M}$, one can define a morphism from $\mathcal{M}$ to $\mathcal{T}_\infty$ which maps $\mathcal{S}_1$ and $\mathcal{S}_2$ into disjoint subtrees of $\mathcal{T}$. This implies that some world above $\mathcal{S}_1$ and $\mathcal{S}_2$ does not fork, and Φ does not have the tree-compatible model property.

In the next section, we study the linear structure $\mathcal{L}_\mathbb{Z}$, which turns out to be a good approximation of paths in trees. The formulas without the tree-compatible model property are discussed in Section IV-B.

A. The closures of linear structures

Now we study the shapes of $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$. We say that the edge $(\underline{i}, \underline{j})$ is *forward* if $i < j$, *backward* if $i > j$, *short* if $|i - j| < 2$, and *long* if $|i - j| \geq 2$. We say that Φ *forces* long (resp. backward) edges if there is a long (resp. backward) edge in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ and that Φ *forces only long forward edges* if it forces long edges but it does not force backward edges.

Definition 9: We say that $\Phi \in \mathsf{UHF}$ satisfies

S1 if Φ does not force long edges,

S2 if Φ forces only long forward edges and there exist $l, a_1, a_2, \ldots, a_l \in \mathbb{N}$ bounded by $\mathfrak{g}(|\Phi|)$ such that for all worlds $\underline{i}$, $\underline{i+b}$, there is an edge from $\underline{i}$ to $\underline{i+b}$ in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ if and only if $b \geq 0$ and $b - 1$ is in the additive closure of $\{a_1, a_2, \ldots, a_l\}$.

S3 if Φ forces long and backward edges and there exists m bounded by $\mathfrak{g}(|\Phi|)$ such that for all worlds $\underline{i}$, $\underline{i+b}$, there is an edge from $\underline{i}$ to $\underline{i+b}$ in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ if and only if m divides $|b-1|$.

Properties S2 and S3 look complicated, so we present a few examples. Below we abbreviate $xRu_1 \wedge u_1Ru_2 \wedge \cdots \wedge u_{i-2}Ru_{i-1} \wedge u_{i-1}Ry$ by xR^iy.

Example 10: Consider a formula $xR^2y \Rightarrow yRx$. Here, Property S3 is satisfied for $m = 3$. For example, $\underline{0}$ is connected to $\underline{1}$, $\underline{4}$, $\underline{7}$ and so on, while $\underline{2}$, $\underline{5}$, $\underline{8}$ and so on are connected to $\underline{0}$ (see Fig. 2a). In general, a formula $xR^iy \Rightarrow yRx$ satisfies Property S3 with $m = i + 1$.

Example 11: Consider a formula $\varphi_3 \wedge \varphi_4$, where $\varphi_i = xR^iy \Rightarrow xRy$. Here, Property S2 is satisfied for $l = 2$, $a_1 = 2$ and $a_2 = 3$. For example, $\underline{0}$ is connected to $\underline{1}$ (as in L_∞), $\underline{3}$ (because of φ_3), $\underline{4}$ (because of φ_4), $\underline{5}$ (because of φ_3, $\underline{0}R\underline{3}$, $\underline{3}R\underline{4}$, and $\underline{4}R\underline{5}$), and so on (see Fig. 2b). In general, for a formula of the form $\varphi_i \wedge \varphi_j$ Property S2 is satisfied with $l = 2$, $a_1 = i - 1$ and $a_2 = j - 1$.

It turns out that Properties S1, S2, and S3 cover all possible formulas.

Lemma 12: Each $\Phi \in \mathsf{UHF}$ satisfies S1, S2, or S3.

Now we show why these linear structures are important. In the tree-compatible case, along each path almost all worlds are connected as in the linear structure. The only exception is for the worlds that are close to the "end" of the model.

Lemma 13: Let $\Phi \in \mathsf{UHF}$, $\mathcal{T}$ be a tree and v_i, v_j be $\mathfrak{g}(|\Phi|)$-inner worlds at the same path. Then there is an edge from v_i to v_j in $\mathfrak{C}_\Phi(\mathcal{T})$ if and only if there is an edge from $h_\mathcal{T}(v_i)$ to $h_\mathcal{T}(v_j)$ in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$.

B. Forks

In this section, we study models of formulas without the tree-compatible model property.

Lemma 14: If $\Phi \in \mathsf{UHF}$ forks at level $\mathfrak{g}(|\Phi|)$, then it has the tree-compatible model property.

This lemma shows that if there is a world that does not fork, then all worlds below some level do not fork. We say that two worlds w, w' of a frame $\mathcal{M}$ are *equivalent* if for each world u we have uRw iff uRw'. Now we argue that if Φ does not fork at the level i, then in structures reachable from worlds at the level i such equivalence is very common.

Lemma 15: Let $\Phi \in \mathsf{UHF}$ be a formula that does not fork, $\mathcal{T}$ be a tree with a bounded degree and w be a world at level $\mathfrak{g}(|\Phi|)$ in $\mathfrak{C}_\Phi(\mathcal{T})$. Then for all i, all the $\mathfrak{g}(|\Phi|)$-followed descendants of w at level $2\mathfrak{g}(|\Phi|) + i$ are equivalent in the frame $\mathfrak{C}_\Phi(\mathcal{T})$.

Example 16: Consider the formula $\Phi = \{\varphi_1, \varphi_2\}$, where $\varphi_1 = xRz \wedge zRy \Rightarrow yRy$ and $\varphi_2 = xRx \wedge xRy \wedge xRz \Rightarrow yRz$, and the tree at the top of Fig. 1. The formula φ_1 enforces the following property: each world that has a predecessor that has a predecessor is reflexive. The formula φ_2 makes the relation R Euclidean except for the non-reflexive worlds. Formula Φ forks at the first two levels.

C. Boundedness

The properties defined above are enough to prove the decidability, but not to obtain the optimal complexity.

We say that a formula Φ is *bounded* if $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ is not a model of Φ, and unbounded otherwise. If the formula is bounded, then there is a k such that the length of each path in each model of Φ is bounded by k, and the value of k depends only on Φ. Recall that in problems $\mathcal{K}_\Phi$-SAT and global $\mathcal{K}_\Phi$-SAT formula Φ is not a part of input. Hence the exact value of k is irrelevant, since it is regarded as a constant.

Now we prove the polynomial model property for bounded formulas. The following argument works for local and global satisfiability.

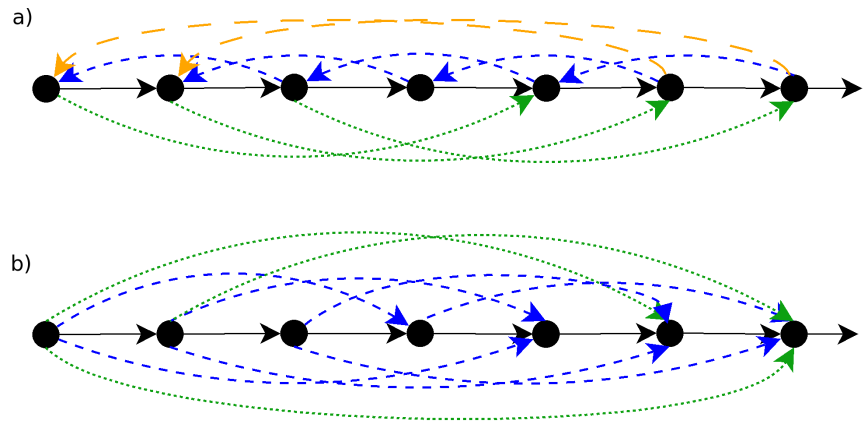

Figure 2. Some closures for the linear structure.

Let Φ be a bounded formula and φ be a modal formula. Then for any model $\mathfrak{M} = \langle W, R, \pi \rangle$ of φ and Φ, we can find a $W' \subseteq W$ such that $\mathfrak{M}_{\restriction W'}$ is a model of φ and $|W'|$ is polynomial in $|\varphi|$. At first, we add an arbitrary world that satisfies φ to W'. Then, recursively, for each world w in W' and each subformula $\Diamond\psi$ of φ, if w has a witness for $\Diamond\psi$ in W but not in W', then we add one such witness to W'. We proceed until a fixed-point is reached. Observe that since the length of each path is bounded by k, then this procedure takes at most k recursive steps, and in each it adds at most $|\varphi|$ worlds for each element of W'. Therefore, at the end we have $|W'| = |\varphi|^k$ and $\mathfrak{M}_{\restriction W'}$ is a model of φ, so indeed we find a polynomial model of φ. Of course, since Φ is universal, $\langle W', R_{\restriction W'} \rangle$ satisfies Φ.

V. Proof of Theorem 1

A well known result shows that every satisfiable modal formula is satisfied in a finite tree. This *tree-model property* is crucial for the robust decidability of modal logics. Standard restrictions of classes of frames lead to similar results, stating that some "nice" models exists for all satisfiable formulas. Here we generalize those results for the classes of models that are definable by the Horn formulas.

For the case of inconsistent Horn formulas, the satisfiability problem is in P (the answer is always "no"), and for the case of consistent bounded formulas the satisfiability is NP-complete — we can simply guess a polynomial model, and the lower bound comes from a trivial reduction from SAT. Below we study consistent and unbounded formulas.

A. Tree-compatible case

Formulas that do not force long edges. Assume that all edges in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ are short. Here we can use standard approaches to satisfiability of modal logic over the class of all models. For local satisfiability we can bound the depth of tree-models and the degree of their worlds linearly in φ and then check the existence of such models in a depth-first search manner in PSpace (see e.g. [13]; please note that while the cited result does not consider reflexivity and symmetry, there are only some minor changes needed to cover these cases).

For global satisfiability we can enforce models of depth exponential with respect to the length of the modal formula φ. The existence of models can be checked by an alternating procedure which first guesses the type of the root and then guesses types of its children and universally repeats the procedure for the children. This algorithm works in alternating polynomial space, and thus the problem is in ExpTime. The corresponding lower bound can be obtained by encoding the halting problem for alternating Turing machine with polynomial space.

Formulas that force only long forward edges. Assume that the condition S2 holds for some $l, a_1, \ldots, a_l$.

This case can be treated similarly to the case of satisfiability over the class of transitive models, i.e. the case of logic K4 (see [13] or Section 6.7 in [12]). Let A be the additive closure of $\{a_1, \ldots, a_l\}$ and c be the product of all positive a_i ($c = \prod_{1 \le i \le l, a_i > 0} a_i$).

Let $P_\mathcal{M}(v)$ be a set of proper k-inner predecessors of v in $\mathcal{M}$ and $W_i = \{\underline{j} | j \le i\}$. We have the following properties.

$$\text{For all } a \in A \text{ and } i, P_{\mathcal{L}_\mathbb{Z}}(\underline{i}) \subseteq P_{\mathcal{L}_\mathbb{Z}}(\underline{i+a}) \quad (1)$$

$$\text{For all } a \in A \text{ and } i, P_{\mathcal{L}_\mathbb{Z}}(\underline{i+a}) \cap W_{i-c} \subseteq P_{\mathcal{L}_\mathbb{Z}}(\underline{i}) \quad (2)$$

For the (1) note that for any $\underline{j} \in P_{\mathcal{L}_\mathbb{Z}}(\underline{i})$ we have $i-j-1 \in A$ and $a \in A$, and therefore $j+a-i-1 \in A$ simply because A is closed under addition. Property (2) follows from property S2 and fact that for each $a \in A$ there exists $a' \in A$ such that $a = a' \mod c$ and $a' < c$ (which follows from Chinese remainder theorem).

$$\text{For } i \ge k, P_{\mathcal{L}_\mathbb{Z}}(\underline{i}) = \bigcup_{a \in A, a < 2c} P_{\mathcal{L}_\mathbb{Z}}(\underline{i-a}) \cup \{\underline{i-1}\} \quad (3)$$

The inclusion "$\supseteq$" comes from property (1). For the "$\subseteq$" case, consider any k-inner predecessor $\underline{j}$ of $\underline{i}$. If $i-j > 2c$, then property (2) for $a = c$ guarantees that $\underline{j} \in P_{\mathcal{L}_\mathbb{Z}}(\underline{i-j})$ only if $\underline{j} \in P_{\mathcal{L}_\mathbb{Z}}(\underline{i})$. If $1 < i-j \le 2c$, then $i-j-1 \in A$. Since $\underline{j}$ is a predecessor of $\underline{j+1}$, $\underline{j} \in P_{\mathcal{L}_\mathbb{Z}}(\underline{j+1}) = P_{\mathcal{L}_\mathbb{Z}}(\underline{i-(i-j-1)})$ and $i-j-1 < 2c$. Case when $i-j = 1$ is trivial.

For a given world w with a type t, we define a *universal requirements* of w, denoted by $UR(w)$, as the subset of t that consists of formulas of the form $\Box\varphi$. Moreover, we define *predecessors requirements* of w, denoted by $PR(w)$, as the set of the universal requirements of the predecessors of w, i.e., $\bigcup\{UR(v)|v$ is a predecessor of $w\}$.

Clearly, property (3) implies that for all $i \geq k$

$$PR(\underline{i}) = \bigcup_{a\in A, a<2c} PR(\underline{i-a}) \cup UR(\underline{i-1}) \cup PR_{ni}(\underline{i}) \quad (4)$$

where $PR_{ni}(wi)$ is a sum of requirements given by those predecessors of $\underline{i}$ that are not k-inner.

Now, we are ready to design an alternating algorithm that guesses a tree-based structure in top-down manner. For input φ, it starts from guessing and verifying first k levels. Then, the algorithm recursively calls procedure $verify(head, URs, PR, \diamond\psi)$ where

- $head$ contains information about the first k levels of structure;
- PRs is a list of predecessors requirements of previous $2c$ k-inner worlds;
- CR is a set of predecessors requirements for the current world;
- $\Diamond\psi$ is a subformula of φ.

The procedure guesses a type t that satisfies ψ and all requirements. Then it guesses a subset of subformulas of φ in order to provide all witnesses for the current world, and for each of them guesses whether they are k-inner. For each witness that is k-inner it simply guesses and verifies the remaining levels (at most $k-1$). For all others witnesses, it universally calls itself for this subformula with PRs and CR updated using Equation (4).

The algorithm described above verifies if φ has a model, but it may run forever. So we add one more parameter to procedure $verify$: a list of visited configurations (i.e. triples $(PRs, CR, \Diamond\psi)$), and additional condition: return "Yes" if the same configuration is visited second time.

It is not hard to see that if this algorithm returns "Yes", then it is possible to build a model. Also, thanks to the property (i) of Lemma 5, if φ has a model, then it has a tree-based model such that all witnesses for the world at the level k are realized at the level $k+1$. In such tree-based model, worlds are connected only if they are on the same path in tree and, moreover, k-inner worlds v, w are connected if and only if $h_{\mathcal{T}}(v)$ and $h_{\mathcal{T}}(w)$ are. Such a canonical model can be guessed and verified by the algorithm. What remain to be explained is that this algorithm works in polynomial time.

The key observation here is that predecessors requirements cannot shrink, i.e., if we have two configurations $(PRs_1, CR_1, \diamond\psi_1)$ and $(PRs_2, CR_2, \diamond\psi_2)$ such that the algorithm visits the second one after the first one, then for each $r \in PRs_1 \cup \{CR_1\}$ (we abuse a notation here since no confusion will result) there is $r' \in PRs_2 \cup \{CR_2\}$ such that $r \subseteq r'$. It means that the number of possible PRs lists along a fixed path can be bounded by $|\varphi|^{2c} \cdot (2c)!$, and the number of all configurations (along a fixed path) can be bounded by $|\varphi|^{2c+1} \cdot (2c)! \cdot |\varphi|$, which is clearly polynomial in $|\varphi|$. Therefore, after a polynomial number of steps some configuration must occur twice. Since APTIME = PSPACE, it leads to the membership is PSPACE in both global and local case.

Formulas that force long and backward edges. We prove that this case is not possible — S3 is inconsistent with the tree-compatible model property.

Let Φ satisfy S3 for some $m > 0$. Let $k = \mathfrak{g}(|\Phi|)$ and $w = 0^k$. By Lemma 13 we see that there are edges from $0^{k+(i+1)(m-1)}$ to $0^{k+i(m-1)}$ in $\mathfrak{C}(\mathcal{T}_\infty)$ for any $i \geq 0$. Define $h : \mathcal{L}_\mathbb{Z} \to \mathfrak{C}(\mathcal{T}_\infty)$ as $h(\underline{x}) = 0^{k-x(m-1)}$ for $x < 0$ and $h(\underline{x}) = 0^k 1^x$ otherwise. Clearly h is an morphism, and by Observation 7 it is also a morphism from $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ to $\mathfrak{C}(\mathcal{T}_\infty)$. Since in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ there is an edge from $\underline{1}$ to $\underline{1-3m+1}$, there is also an edge from $0^k 1$ to $0^{k+1-3m+1}$ and therefore w is not forking.

B. The tree-incompatibility

Let Φ be a formula without the tree-compatible model property. We start with the observation that says that if we have two equivalent worlds with the same types, then we can remove one of them.

Observation 17: Let $\mathfrak{M} = \langle W, R, \pi\rangle$ be a structure such that $\langle W, R\rangle \models \Phi$, and φ be a modal formula such that $\mathfrak{M} \models \varphi$. Let $W_e \subseteq W$ be a set of equivalent worlds in $\mathfrak{M}$ and $w \in W_e$. If for all $\psi \in tp^{\varphi}_{\mathfrak{M}}(w)$ there exists $w' \in W_e \setminus \{w\}$ such that $\psi \in tp^{\varphi}_{\mathfrak{M}}(w')$, then for $W' = W \setminus \{\underline{1}\}$ we have $\langle W', R_{\restriction W'}\rangle \models \Phi$ and $\mathfrak{M}_{\restriction W'} \models \varphi$.

The proof is straightforward — the types of remaining worlds do not change.

Let $\mathfrak{M}$ be a tree-based model based on the frame $\mathfrak{C}(\mathcal{T})$. We denote by *level i of* $\mathfrak{M}$ the set of worlds from $\mathfrak{M}$ such that the length of the path from root to w in $\mathcal{T}$ (notice that $\mathcal{T}$ is a tree) is equal i.

Observation 18: Let φ be a formula and $\mathfrak{M}$ be a tree-based model of Φ and φ. Then there is a model of Φ and φ such that the size of each level of $\mathfrak{M}$ is bounded polynomially in $|\varphi|$.

First, observe that the number of worlds at level $i \leq 2\mathfrak{g}(|\Phi|)$ can be bounded by $|\varphi|^i$ because Lemma 5 guarantees that the degree of the tree is bounded by $|\varphi|$. Thanks to Lemma 14, Φ does not fork at level $\mathfrak{g}(|\Phi|)$. It follows from Lemma 15 that for all worlds w at the level $\mathfrak{g}(|\Phi|)$ and all $i \geq 2\mathfrak{g}(|\Phi|)$, all descendants of w at the level i are equivalent. Therefore we can remove all but $|\varphi|$ of them. Since the number of worlds at the level $\mathfrak{g}(|\Phi|)$ can be bounded by $|\varphi|^{\mathfrak{g}(|\Phi|)}$, the number of worlds at the level $i > 2\mathfrak{g}(|\Phi|)$ can be bounded by $|\varphi|^{\mathfrak{g}(|\Phi|)} \cdot |\varphi|$, so polynomially in $|\varphi|$.

Observation 18 says that we can reduce the number of worlds needed at each level by some polynomial of $|\varphi|$. The existence of such models can be verified by a non-deterministic machine working in polynomial space that first guesses first $2\mathfrak{g}(|\Phi|)$ levels, and then recursively guesses and verifies the consecutive levels, similarly to the tree-compatible case. Since the number of worlds needed at each level can be bounded polynomially in $|\varphi|$, such an algorithm would work in NPSPACE=PSPACE [14]. We can conclude that here the satisfiability problem is in PSPACE. This ends the proof of Theorem 1. However, it does not lead to the optimal complexity.

VI. Sharpening the complexity

In this section, we study the satisfiability problems more carefully to obtain the precise complexity. The complexity will by summarized in Table I.

A. Formulas with TCMP

Proposition 19: For a given UHF formula Φ, if Φ has the tree-compatible model property and satisfies S2, then global $\mathcal{K}_\Phi$-SAT is in NP.

Proof: Let Φ satisfy S2 for some $l, a_1, \dots, a_l$ bounded by $\mathfrak{g}(|\Phi|)$ and let c be the product of all a_i and $\mathfrak{M}$ be a $\mathcal{T}$-based model of φ from $\mathcal{K}_\Phi$. We prove that φ has a model satisfying Φ with the number of types bounded by $|\varphi| \cdot c$.

We say that a world w at the level i (of $\mathcal{T}$) is *saturated* if for all k and every successors w' of w at levels $i + kc$, $PR(w) = PR(w')$.

Observe that in $\mathfrak{M}$ there is a world w such that the subtree rooted in w contains only saturated worlds. Let $\mathfrak{M}'$ be this subtree. Of course, $\mathfrak{M}'$ is a $\mathcal{K}_\Phi$-model of φ. For each subformula $\Diamond\psi$ of φ and each $i < c$, if there is a world in $\mathfrak{M}'$ at level $jc + i$ for some j that satisfies ψ, then we take a 1-type of such a world and call it $t_{\psi,j}$. It is not hard to see that there exists a model $\mathfrak{M}''$ that contains only w and worlds of this types — we can construct such a model starting from w, and then recursively constructing new levels that contains all needed witnesses for the previous level.

The non-deterministic algorithm proceeds as follows. First, it guesses sets of requirements PR_0, PR_1, $\dots$, PR_{c-1}, and a subset of types of the form $t_{\psi,j}$. If this types are consistent with requirements and for each $t_{\psi,i}$ we can find $t_{\psi_1,i+1 \mod c}, \dots, t_{\psi_s,i+1 \mod c}$ such that these types provides all needed witnesses for a world of type $t_{\psi,i}$, then it returns "Yes", otherwise it returns "No". Clearly, it works in polynomial time and solves global $\mathcal{K}_\Phi$-SAT. ∎

Proposition 20: For a given UHF formula Φ, if Φ has the tree-compatible model property, then $\mathcal{K}_\Phi$-SAT is PSPACE-hard.

This proposition can be proved by reducing the QBF problem, adjusting the usual technique (see e.g. [13]).

B. Formulas without TCMP that do not force long edges

Proposition 21: For a given UHF formula Φ, if Φ does not have the tree-compatible model property and satisfies S1, then it has a polynomial model property for the local satisfiability problem.

Proposition 21 follows from the fact that in the local satisfiability case for any tree-based model $\mathfrak{M}$ based of $\mathfrak{C}(\mathcal{T})$ such that of $\mathfrak{M}_0, \underline{0} \models \varphi$, we can simply remove all worlds w that are at the levels greater than d, the quantifier depth of φ. Indeed, S1 says that there are only short edges in closures and therefore the removed worlds were not reachable by φ. The resulting model contains at most $|\varphi|^{2\mathfrak{g}(|\Phi|)+1}$ worlds at first $2\mathfrak{g}(|\Phi|)$ levels and then at most $|\varphi|$ worlds at each of remaining $d - 2\mathfrak{g}(|\Phi|)$ levels, so clearly a polynomial number of worlds.

Proposition 22: For a given UHF formula Φ, if Φ does not have the tree-compatible model property and satisfies S1, then global $\mathcal{K}_\Phi$-SAT is PSPACE-hard.

Proof: To make the proof more readable, we consider only the formula $\Phi = \{sRt \wedge tRy \wedge sRx \Rightarrow xRy\}$. Proofs for other cases are similar.

A *domino system* is a tuple $\mathcal{D} = (D, D_H, D_V)$, where D is a set of domino pieces and $D_H, D_V \subseteq D \times D$ are binary relations specifying admissible horizontal and vertical adjacencies. The bounded-space domino problem is defined as follows. For a given triple $\langle D, V_\mathcal{D}, H_\mathcal{D}\rangle$, where $V_\mathcal{D}, H_\mathcal{D} \subseteq D \times D$, and $n = |D|$, is there a tiling $t : \mathbb{Z}_n \times \mathbb{N} \to D$ such that for all $k < n$ and $l \in \mathbb{N}$, $(t(k,l), t(k,l+1)) \in V_\mathcal{D}$ and if $k < n-1$, then $(t(k,l), t(k+1,l)) \in H_\mathcal{D}$? It is well-known that this problem is PSPACE-complete.

Let $\langle D, V_\mathcal{D}, H_\mathcal{D}\rangle$ be an instance of the bounded-space domino problem. We define a formula $\varphi = \psi_c \wedge \psi_v \wedge \psi_h \wedge \psi_e$ over variables $\{t_0, \dots, t_{n-1}\} \cup D$ where:

- $\psi_c = \bigvee_{d\in D} d \wedge \bigwedge_{d,d'\in D, d\neq d'} (\neg d \vee \neg d')$;
- $\psi_e = \bigwedge_{i<n} \Diamond t_i$;
- $\psi_v = \bigwedge_{i<n} \bigwedge_{d\in D} (t_i \wedge d \Rightarrow (\bigvee_{(d,d')\in V_\mathcal{D}} \Box(t_i \Rightarrow d')))$;
- $\psi_h = \bigwedge_{i<n-1} \bigwedge_{d\in D} (\Box(t_i \wedge d) \Rightarrow (\bigvee_{(d,d')\in H_\mathcal{D}} \Box(t_{i+1} \Rightarrow d')))$.

Clearly, the reduction is polynomial. Suppose that $\mathfrak{M}$ is a model of Φ and φ and v_0 is any world of $\mathfrak{M}$. We define tiling t by repeating the following procedure. For a given i, we define $v_{j,i}$ as a successor of v_i that satisfies t_j and we put $t(j,i) = d$, where d is satisfied in $v_{j,i}$. Note that ψ_e guarantees that such a successor exists, ψ_v guarantees that if there is more than one such successor, then all of them satisfy the same d, and ψ_c guarantees that all worlds satisfy precisely one d. Finally, we set v_{i+1} equal to any successor of v_i that satisfies t_0.

It is not hard to see that for all $k < n-1$ and $l \in \mathbb{N}$ property $(t(k,l), t(k+1,l)) \in H_\mathcal{D}$ is guaranteed by ψ_h since both $v_{k,l}$ and $v_{k+1,l}$ are successors of v_l. To check the other property, consider any $l \in \mathbb{N}$ and $k < n$. Since $v_l R v_{l+1}$, $v_{l+1} R v_{k,l+1}$, and $v_l R v_{k,l}$, Φ guarantees that we have $v_{k,l} R v_{k,l+1}$ and therefore ψ_v guarantees that $(t(k,l), t(k,l+1)) \in V_\mathcal{D}$.

We showed that if φ has a model that satisfies Φ, then the domino problem has a solution. It should be now easy to see that the converse is also true. ∎

C. Formulas without TCMP that force only long forward edges

Proposition 23: For a given UHF formula Φ, if Φ does not have the tree-compatible model property and satisfies S2, then global and local $\mathcal{K}_\Phi$-SAT are NP-complete.

Proof: Let Φ be a UHF formula that does not have the tree-compatible model property and satisfies S2 for some $l, a_1, \dots, a_l$, φ be a modal formula and $\mathfrak{M}$ be a tree-based model of Φ and φ. Let $c = a_1 \cdot \dots \cdot a_l$ and for a world w at level $\mathfrak{g}(|\Phi|)$ and $i > \mathfrak{g}(|\Phi|)$, set C_i^w be the set of all descendants of w at level i. According to previous observations we may

Properties of Φ	global-$\mathcal{K}_\Phi$-SAT	$\mathcal{K}_\Phi$-SAT
Φ is inconsistent	P	P
Φ is consistent and bounded	NP-c	NP-c
Φ is consistent, unbounded, …		
…has the TCMP and satisfies S1	EXPTIME-c	PSPACE-c
…has the TCMP and satisfies S2	NP-c	PSPACE-c
…has the TCMP and satisfies S3	impossible	
…and does not have the TCMP and satisfies S1	PSPACE-c	NP-c
…and does not have the TCMP and satisfies S2	NP-c	NP-c
…and does not have the TCMP and satisfies S3	NP-c	NP-c

Table I
A SUMMARY OF A COMPLEXITY OF A SATISFIABILITY PROBLEM FOR MODAL LOGIC DEFINED BY HORN FORMULAS.

assume that the size of each such set is polynomial in $|\varphi|$. Our goal is to show that for any w, it is enough to consider only polynomially many non-isomorphic sets C_i^w. Clearly, it will make the algorithm described above run in the polynomial time.

In Section V-A we showed similar property, but the technique used there is not sufficient for this case — now, it is not enough just to satisfy one formula of the form $\Diamond\psi$ at each level. We solve this problem in the following way: in each C_i^w, we put as many witnesses as possible. We extend the notation from Section V-A defining $PR(X) = \bigcup_{w\in X} PR(w)$. Note that since all worlds in C_i^w are equivalent, for any $v \in C_i^w$ we have $PR(v) = PR(C_i^w)$. Moreover, Properties (1) and (2) also holds in this case.

Observation 24: Let w, v be worlds such that $v \in C_j^w$ for some j and let i be such that $\mathfrak{g}(|\Phi|) < i < j$ and c divides $j - i$. If $UR(v) \subseteq PR(C_{i+1}^w)$ and $PR(v) = PR(C_i^w)$, then model obtained by adding a copy v' of v to C_i^w satisfies both Φ and φ.

Note that the set of successors of v is a subset of the set of successors of v', and therefore v has all needed witnesses. Moreover, the set of predecessors of v' is a subset of the set of predecessors of v, so v' does not violate any predecessor requirements. Finally, v' does not add any new requirements, it should be clear that new model satisfies φ. Therefore the new model satisfies φ and, in an obvious way, Φ.

Observation 25: Let w be a world at level $\mathfrak{g}(|\Phi|)$, let $i > \mathfrak{g}(|\Phi|)$, and let $A = \{0, 1, \dots\}$ be a (possibly finite) set of consecutive numbers. Let $\mathcal{C} = \bigcup\{C_{i+ac}^w | a \in A\}$ be such that for all $j, j' \in A$, $PR(C_j^w) = PR(C_{j'}^w)$ and $PR(C_{j+1}^w) = PR(C_{j'+1}^w)$. Then, we can define a set C' with $|C'| \leq |\varphi|$ such that each element of $\bigcup\mathcal{C}$ can be replaced by a copy of an element from C' in a way such that the obtained model is still a model of φ and Φ.

Let $C = \bigcup\mathcal{C}$. We define a $C' \subseteq C$ in the following way. For every subformula of φ of the form $\Diamond\psi$, if there is a type t satisfying ψ such that t is realized in infinitely many elements of $\mathcal{C}$, then we take one world of this type and add it to C'. If there is no such type, but there is a world in C that satisfies ψ, then we find a maximal $j \in A$ such that there is such a world $v \in C_{i+jc}^w$ and we add v to C'. Clearly, $|C'| \leq |\varphi|$. Then, we define $C'^{\underline{i+jc}} = C' \cap \bigcup_{a\in A, a\geq j} C_{i+ac}^w$ and replace each C_{i+jc}^w by $C'^{\underline{i+jc}}$. Note that such a model satisfies both φ and ϕ.

Let w be a world at level $\mathfrak{g}(|\Phi|)$ and i be such that $\mathfrak{g}(|\Phi|) \leq i < \mathfrak{g}(|\Phi|) + c$. property (1) still holds and shows that the sequence $PR(C_i^w), PR(C_{i+c}^w), PR(C_{i+2c}^w)\dots$ never shrinks, and the same holds for $PR(C_{i+1}^w), PR(C_{i+c+1}^w), PR(C_{i+2c+1}^w)\dots$. Therefore, the sequence $C_i^w, C_{i+c}^w, C_{i+2c}^w$ can be split into at most $|\varphi|^2$ subsequences that satisfy the requirements of Observation 25, so the number of different sets of the form C_i^w can be bounded by $|\varphi|^3$. Taking into account all possible w and i, we can bound the number of possible sets C_i^w by $|\varphi|^{\mathfrak{g}(|\Phi|)} \cdot c \cdot |\varphi|^3$, which is clearly polynomial in φ. ■

D. Formulas without TCMP that force long and backward edges

Proposition 26: For a given UHF formula Φ, if Φ does not have the tree-compatible model property and satisfies S3, then it has a polynomial model property.

Suppose that Φ does not have the tree-compatible model property and satisfies S3 for some k, m. Observe that in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ for all $i > k$ and $l \geq 0$, worlds $\underline{i}$ and $\underline{i + lm}$ are equivalent. Let $\mathfrak{M}$ be a model of φ. It follows from Lemmas 14 and 15 that for all w at the level $\mathfrak{g}(|\Phi|)$ and all i, all descendants of w at levels $2\mathfrak{g}(|\Phi|) + i$, $2\mathfrak{g}(|\Phi|) + i + m$, $2\mathfrak{g}(|\Phi|) + i + 2m$, …are equivalent. We can remove all but polynomially many of them and obtain a smaller model that still satisfies φ. We may repeat this procedure for all such w, finally obtaining model of polynomial size in $|\varphi|$.

VII. HORN FORMULAS AND THE EQUALITY

In this section, we prove Theorem 2. It is not hard to see that each negative occurrence of equality may be eliminated by simply identifying variables. Thus, in the rest of this section we focus on formulas without negative occurrences of equality. For a given $\Phi \in \mathsf{UHF}^=$, let $\Phi^\#$ contain all the clauses of Φ except for those with the positive occurrence of equality. We define three properties of $\mathsf{UHF}^=$ formulas.

Properties of Φ	global-$\mathcal{K}_\Phi$-SAT	$\mathcal{K}_\Phi$-SAT
Satisfies E1	The same as for $\Phi^{\#}$	
Satisfies E2	NP-c	NP-c
Satisfies E3	NP-c	NP-c

Table II
A SUMMARY OF A COMPLEXITY OF A SATISFIABILITY PROBLEM FOR MODAL LOGIC DEFINED BY CONSISTENT HORN FORMULAS WITH EQUALITY.

Definition 27: We say that $\Phi \in \mathsf{UHF}^{=}$ satisfies

E1 if for each tree $\mathcal{T}$, $\mathfrak{C}_{\Phi^{\#}}(\mathcal{T})$ satisfies Φ.

E2 if there is a tree $\mathcal{T}$ and two worlds w, v at different levels in $\mathcal{T}$ such that some clause of Φ of the form $\Psi \Rightarrow x = y$ is not satisfied in $\mathfrak{C}_{\Phi^{\#}}(\mathcal{T})$ for some instantiation that substitute x with w and y with v.

E3 if it does not satisfy E2 and there is a tree $\mathcal{T}$ and two worlds w, v at the same level in $\mathcal{T}$ such that some clause of Φ of the form $\Psi \Rightarrow x = y$ is not satisfied in $\mathfrak{C}_{\Phi^{\#}}(\mathcal{T})$ for some instantiation that substitute x with w and y with v.

A quick check shows that each formula satisfy E1, E2, or E3. It is also not hard to see that if a formula satisfies E1, then a modal formula has a model based on a frame of $\mathcal{K}_{\Phi^{\#}}$ if and only if it has a model based on some frame from $\mathcal{K}_\Phi$. Since $\Phi^{\#}$ is UHF formula, we can simple apply previous result. The remaining two cases are more interesting.

If Property E2 holds, then let $\mathcal{T}$ be a tree and w, v be worlds at different levels in $\mathcal{T}$ such that for some clause $\Psi \Rightarrow x = y$ we have $\Psi(w, v)$. Consider the morphism $h_\mathcal{T}$ and worlds $h_\mathcal{T}(w)$ and $h_\mathcal{T}(v)$. Since $\Phi \in \mathsf{UHF}$, we may simply apply Observation 7 to verify that in $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ we have $\Psi(h_\mathcal{T}(w), h_\mathcal{T}(v))$. By the definition of $h_\mathcal{T}$ we know that $h_\mathcal{T}(w) \neq h_\mathcal{T}(v)$, and therefore $\mathfrak{C}_\Phi(\mathcal{L}_\mathbb{Z})$ is not a model of Φ. It is easy to verify that this imply that Φ is bounded and therefore it has the polynomial model property.

For the case when Φ satisfies E3 we need the following definition. We say that (possibly infinite) directed acyclic graph (DAG) is proper if it has a root r such that all vertices of this DAG are reachable from r, and for all elements v, v' all path from v to v' have the same length. Clearly, trees are special cases of proper DAGs.

Now we can adjust Lemma 5: if φ is $\mathcal{K}_\Phi$-satisfiable, then there exists a proper directed acyclic graph $\mathcal{T}$ with the degree bounded by $|\varphi|$ and a labeling $\pi_\mathcal{T}$, such that $\langle \mathcal{T}, \pi_\mathcal{T}\rangle$ is a model of φ and $\langle \mathfrak{C}_\Phi(\mathcal{T}), \pi_\mathcal{T}\rangle$ is a model of φ that satisfies Φ.

As for trees, we define the level of v in DAG as the length of path from the root to v. Therefore, morphism $\pi_\mathcal{T}$ is well-defined also for DAGs. Then, we adjust Lemma 15: if, for any proper DAG $\mathcal{T}$, w is a world at level $\mathfrak{g}(|\Phi|)$ in $\mathfrak{C}_\Phi(\mathcal{T})$, then for all i, there is at most one $\mathfrak{g}(|\Phi|)$-followed descendant of w at level $2 \cdot \mathfrak{g}(|\Phi|) + i$ in the frame $\mathcal{T}$. It means that the number of worlds at each level can be bounded by $|\varphi|^{\mathfrak{g}(|\Phi|)}$. Then we show, as in the case of tree-incompatibility, that it is enough to consider polynomial number of different types of levels, and therefore that both the global and the local satisfiability problems are NP-complete.

VIII. INEQUALITY LEADS TO THE UNDECIDABILITY

In this section we prove Theorem 3. We work with signatures consisting of a single binary symbol R, and a number of unary symbols, including P_{ij}, A_{ij}, and e_i for $0 \leq i, j \leq 2$. Structures over such signatures can be naturally viewed as Kripke structures in which R is the accessibility relation, and unary relations describe valuations of propositional variables.

First, we define a formula Γ and we prove that global $\mathcal{K}_\Gamma$-SAT is undecidable. Then we adapt the technique from [4] to show that also local $\mathcal{K}_{\Gamma'}$-SAT is undecidable, for a formula Γ' being a simple modification of Γ.

In the proof of the undecidability, we use the inequality only to say that the out-degree of a vertex is large. That is, we define an abbreviation $deg_{\geq k}(v)$ that uses the fresh variables $u_1^v, \ldots, u_k^v$ as follows.

$$deg_{\geq k}(v) = \bigwedge_{1 \leq i \leq k} (vRu_i^v) \wedge \bigwedge_{1 \leq i < j \leq k} u_i^v \neq u_j^v$$

For example, the formula $deg_{\geq 5}(v) \Rightarrow vRz$ says that all the worlds with out-degree greater than five are connected to all worlds.

Now, we the formula Γ that gives us undecidability as

$$xRy \wedge xRu \wedge uRz \wedge deg_{\geq 2}(x) \wedge deg_{\geq 4}(u) \wedge deg_{\geq 2}(z) \Rightarrow yRz$$

The formula Γ contains only one Horn clause. Note that the structure $\mathfrak{G}_\mathbb{N}$ illustrated in Fig. 3 is a model of Γ.

The idea of the proof is similar to the proof of the undecidability presented in [5]. In both cases, we show how to enforce that models are extensions of the standard grid and then we encode a domino system in it. Below we show how to use the modal logic to define a grid-like structure in $\mathcal{K}_\Gamma$. For the details of encoding the domino systems in grids the reader is referred to [5]. Notice that both proofs work also for the finite satisfiability problem, i.e. satisfiability over the class of finite structures from $\mathcal{K}_\Gamma$.

To get the undecidability we construct a modal formula τ such that any model $\mathfrak{M} \models \tau$ from $\mathcal{K}_\Gamma$ locally looks like a grid. Namely, τ says that:

(i) each element is labeled with exactly one of the variables from the set $\{P_{ij} | i, j \in \{0, 1, 2\}\} \cup \{A_{ij} | i, j \in \{0, 1, 2\}\} \cup \{e_{ij}^k | i, j, k \in \{0, 1, 2\}\}$.

(ii) every element satisfying P_{ij} has three R-successors: one in $P_{(i+1 \mod 3)j}$, one in $P_{i(j+1 \mod 3)}$, and one in A_{ij};

(iii) every element satisfying A_{ij} has four successors: one in $P_{(i+1 \mod 3)(j+1 \mod 3)}$, one in e_{ij}^0, one in e_{ij}^1, and one in e_{ij}^2

(iv) every element satisfying e_{ij}^k has a successor satisfying A_{ij}.

All those properties are easy to express in modal logic. Observe that each model of this formula contains a world satisfying P_{00}. If we consider now any world a_x satisfying,

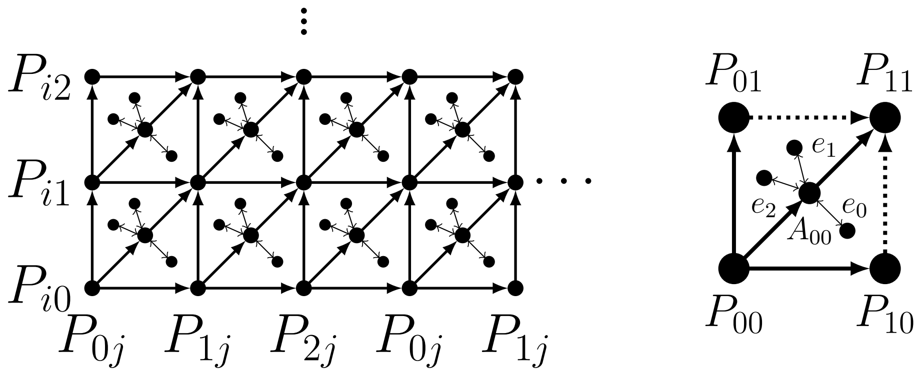

Figure 3. The structure $\mathfrak{G}_{\mathbb{N}}$. Its universe is $\mathbb{N} \times \mathbb{N}$.

e.g., P_{00} in a model, we see that the property (ii) of τ enforces the existence of its horizontal successor a_y satisfying P_{10}, its vertical successor $a_{y'}$ satisfying P_{01} and its upper-right diagonal successor a_u satisfying A_{00} (see Fig. 3). By (iii), the element satisfying A_{00} has four successors, including a_z satisfying P_{11}. It should be clear that when we instantiate Γ with the worlds a_x, a_y (or $a_{y'}$), a_u, and a_z, then the antecedent of the Γ is satisfied, and that implies the edges from a_y and a'_y to a_z (see Fig. 3, dotted edges).

Local satisfiability Observe that the trick from [4], that reduces the local satisfiability problem to the global one, requires a formula which is not a Horn formula (one of its parts is of the form $\neg xRx \Rightarrow xRy$), so we cannot use it. It turns out, however, that only a slight modification is needed. Observe that our proof of the undecidability of global satisfiability over $\mathcal{K}_\Gamma$ works for the subclass of models such that the out-degree of each world is bounded by four. Now, we enforce by a modal formula the existence of a world with out-degree 5 and, by a first-order formula, we make it connected to all worlds. Such a *universal world* can be then used to reach all relevant elements in the model.

$$\Gamma' = (deg_{\geq 5}(u) \wedge u \neq v \Rightarrow uRv) \wedge \Gamma$$

In the modal formula we use a fresh symbols $f_1, \ldots, f_5$ to guarantee that a world with the degree at least 5 exists. Now, for each modal formula φ we define its local version φ^l by $\bigwedge_{i \in \{1,\ldots,5\}} \Diamond f_i \wedge \bigwedge_{1 \leq i < j \leq 5} \neg\Diamond(f_i \wedge f_j) \wedge \Box\varphi$ such that φ^l is locally satisfiable over $\mathcal{K}_{\Gamma'}$ iff φ is globally (finitely) satisfiable over $\mathcal{K}_\Gamma$. This ends the proof of Theorem 3.

See subsection 5.6 of [4] for the details of the outlined trick.

IX. Future work

In this paper, we focused on the case when the first-order formula is fixed. However, the question about the precise complexity of the satisfiability problem when both formulas are parts of instances is also interesting. Is this problem in PSPACE for the case of local satisfiability?

The ultimate aim of our research is to give the complete characterization of the decidability of the elementary modal logics. One of possible solution would be to prove the decidability of the following problem. For a given (universal) first-order formula Φ, is $\mathcal{K}_\Phi$-SAT decidable?

One more natural question concerns other classes of formulas that lead to decidable logic. For instance, in [4] the question about the class of the universal first-order formulas that imply transitivity was stated. It can be easily shown that the finite satisfiability for such formulas is decidable, but the general solution remains unknown.

References

[1] E. Hemaspaandra and H. Schnoor, "On the complexity of elementary modal logics," in *STACS*, ser. LIPIcs, vol. 1. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, Germany, 2008, pp. 349–360.

[2] H. Sahlqvist, "Completeness and correspondence in the first and second order semantics for modal logic," *Proceedings of the Third Scandinavian Logic Symposium*, 1973.

[3] E. Hemaspaandra, "The price of universality," *Notre Dame Journal of Formal Logic*, vol. 37, no. 2, pp. 174–203, 1996.

[4] E. Hemaspaandra and H. Schnoor, "A universally defined undecidable unimodal logic," in *MFCS*, ser. Lecture Notes in Computer Science, vol. 6907. Springer, 2011, pp. 364–375.

[5] E. Kieroński, J. Michaliszyn, and J. Otop, "Modal logics definable by universal three-variable formulas," in *FSTTCS*, ser. LIPIcs, vol. 13. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2011, pp. 264–275.

[6] M. Mortimer, "On languages with two variables," *Mathematical Logic Quarterly*, vol. 21, no. 1, pp. 135–140, 1975.

[7] M. Otto, "Two variable first-order logic over ordered domains," *Journal of Symbolic Logic*, vol. 66, pp. 685–702, 1998.

[8] E. Kieroński, J. Michaliszyn, I. Pratt-Hartmann, and L. Tendera, "Extending two-variable first-order logic with equivalence closure," in *LICS '12: Proceedings of the 29th IEEE symposium on Logic in Computer Science*, 2012.

[9] E. Grädel, "On the restraining power of guards," *J. Symbolic Logic*, vol. 64, pp. 1719–1742, 1999.

[10] E. Grädel and I. Walukiewicz, "Guarded fixed point logic," in *Fourteenth Annual IEEE Symposium on Logic in Computer Science*, 1999, pp. 45–54.

[11] J. Michaliszyn, "Decidability of the guarded fragment with the transitive closure," in *ICALP (2)*, 2009, pp. 261–272.

[12] P. Blackburn, M. de Rijke, and Y. Venema, *Modal Logic*, ser. Cambridge Tracts in Theoretical Computer Scie. Cambridge: Cambridge University Press, 2001, vol. 53.

[13] R. E. Ladner, "The computational complexity of provability in systems of modal propositional logic," *Siam Journal on Computing*, vol. 6, pp. 467–480, 1977.

[14] W. J. Savitch, "Relationships between nondeterministic and deterministic tape complexities," *J. Comput. Syst. Sci.*, vol. 4, pp. 177–192, April 1970.