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A Relation between the Comparability Graph and the Number of Linear Extensions*

GRZEGORZ STACHOWIAK
Institute of Computer Science, University of Wrocław, Przesmyckiego 20, 51-151 Wrocław, Poland

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Abstract. In this note, we prove that the comparability graph of a poset P has less edges than that of a poset Q on the same set of elements, then P has more linear extensions than Q. This solves a problem posed by Kelly [1].

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Let P be a finite partially ordered set (simply called a poset). We denote by $G(P)$ its comparability graph, i.e. the vertex set of $G(P)$ is P and two vertices are adjacent if they are comparable in P.

For two graphs G and H we write $G = H$ if G and H are isomorphic and $G \subseteq H$ if the graphs are on the same vertex set and the set of edges of G is included in that of H.

A total order L of elements of P is a linear extension of P if and only if $x \leqslant y$ in P implies $x \leqslant y$ in L for any $x, y \in P$. We denote by $e(P)$ the number of all linear extensions of P.

It is known [2], that the number of linear extensions of a poset is a comparability invariant, i.e. if $G(P) = G(Q)$, then $e(P) = e(Q)$. Our main result may be considered as a generalisation of this fact.

THEOREM. *If P and Q are two posets on the same set of elements and $G(P) \subseteq G(Q)$ then $e(P) \geqslant e(Q)$ and $e(P) = e(Q)$ implies $G(P) = G(Q)$.*

It is evident that if I is the set of all minimal elements in a poset P then

$$e(P) = \sum_{x \in I} e(P - x). \qquad (*)$$

We will need the following Lemma.

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LEMMA. *If A is an arbitrary antichain in a poset P then*

$$\sum_{a \in A} e(P - a) \leqslant e(P).$$

Proof. We proceed by induction on $n = |P|$.

(1) If $n = 2$ then the Lemma holds by inspection of all possible posets and antichains.

(2) If $n > 2$ then we assume that the Lemma holds for all $(n-1)$-element posets. Assume, that P is a poset on n elements and I denotes the set of its minimal elements. Let I_y denote the subset of P consisting of those elements which become minimal when the element y is removed from P.

(2.1) By the inductive hypothesis, for each $x \in I$ we have:

(2.1.1) If $x \notin A$, then applying the Lemma for $P' = P - x$ and $A' = A$ we obtain

$$\sum_{a \in A} e(P - x - a) \leqslant e(P - x).$$

(2.1.2) If $x \in A$, then applying the Lemma for $P' = P - x$ and $A' = A \cup I_x - x$ we obtain

$$\sum_{a \in A - x} e(P - x - a) + \sum_{a \in I_x} e(P - x - a) \leqslant e(P - x).$$

(2.2) By (*) we have for all $a \in A$:

(2.2.1) If $a \notin I$, then

$$\sum_{x \in I} e(P - x - a) = e(P - a).$$

(2.2.2) If $a \in I$, then

$$\sum_{x \in I - a} e(P - x - a) + \sum_{x \in I_a} e(P - x - a) = e(P - a).$$

(2.3) Summing up all inequalities from 2.1 we obtain

$$\sum_{x \in I} \sum_{a \in A - x} e(P - x - \alpha) + \sum_{x \in I \cap A} \sum_{a \in I_a} e(P - x - \alpha) \leqslant \sum_{x \in I} e(P - x)$$

or equivalently

$$\sum_{a \in A} \sum_{x \in I - a} e(P - x - \alpha) + \sum_{a \in I \cap A} \sum_{x \in I_x} e(P - x - \alpha) \leqslant e(P).$$

Applying 2.2 we obtain

$$\sum_{a \in A} e(P - a) \leqslant e(P).$$

□

Proof of the Theorem. We proceed by induction on $n = |P| = |Q|$.

(1) If P and Q are one-element posets then the Theorem is obvious.

(2) If $n > 1$ then we assume that the Theorem holds for all $(n-1)$-element posets. Let A be the set of minimal elements of Q.

Proof of the Theorem. We proceed by induction on $n = |P| = |Q|$.

(1) If P and Q are one-element posets then the Theorem is obvious.

(2) If $n > 1$ then we assume that the Theorem holds for all $(n-1)$-element posets. Let A be the set of minimal elements of Q.

We have

(i) $e(Q) = \sum_{a \in A} e(Q - a)$,

(ii) $\sum_{a \in A} e(Q - a) \leqslant \sum_{a \in A} e(P - a)$,

(iii) $\sum_{a \in A} e(P - a) \leqslant e(P)$.

The equality (i) follows from (*), (ii) follows from the inductive hypothesis and (iii) follows from the Lemma.

Now we prove, that if $G(Q)$ has more edges than $G(P)$, then this inequality is sharp. We consider two cases

(2.1) $|A| > 1$ or $A = \{a\}$ and there exists an edge in $Q - a$ which is absent in P. In both cases there exists $a \in A$, such that this edge belongs to $Q - a$. Then inequality (ii) is sharp.

(2.2) $A = \{a\}$ and there exists an edge $[a, b]$ in Q, which does not belong to P. Thus $\{a, b\}$ is an antichain in P and by the Lemma

$$e(P - a) < e(P - a) + e(P - b) \leqslant e(P). \qquad \square$$

Hence, inequality (iii) is sharp.

In connection with the fact, that the number of linear extensions of a poset is a comparability invariant, we can define an invariant $E(G)$ for a graph G, which is equal to $e(P)$ if $G = G(P)$. By the Lemma, $E(G)$ can be defined as follows

(1) If G is an one-vertex graph then $E(G) = 1$.

(2) Otherwise,

$$E(G) = \max_{A \in \mathbf{A}} \sum_{a \in A} E(G - a),$$

where $\mathbf{A}$ is the collection of all independent sets of G, i.e. such subsets of vertices of G no two of which are adjacent.

If G is a comparability graph of a poset P, then by the Lemma we have $E(G) \leqslant e(P)$ and by (*) $- E(G) \geqslant e(P)$, therefore $E(G) = e(P)$.

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