Finding Parity Difference by Involutions

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Abstract

Parity difference equal to 0 or ± 1 is a necessary condition for the existence of minimal change generation algorithms for many combinatorial objects. We prove, that finding parity difference for linear extensions of posets is $\#P$-complete. We also show a new method of finding parity difference for strings representing forests and a combinatorial interpretation of this result as well as all cases when this value is equal 0 or ± 1 (see [5]).

1 Introduction

We have given a set of combinatorial objects. The process of listing these combinatorial objects such that each object is listed exactly once we call *generation* of these combinatorial objects. Generation by minimal changes consists in obtaining the next generated object by some minimal change of the current object. Minimal change is not a precise notion. In each case we have to define it for a particular kind of objects. Well known algorithms for generation of 0-1 strings of given length n by making change at only one position to get the next string of the generated sequence [1] and for generation n-element permutations by adjacent transpositions [4] [12] [13] are examples of minimal change generation algorithms that can be found in many textbooks on combinatorial algorithms.

Such algorithms often have faster implementations than other types of generation algorithms.

A way to understand how to construct such algorithms is by defining a *graph of minimal changes* $\mathcal{G}$. The vertex set $\mathcal{V}$ of this graph consists of combinatorial objects that are to be generated, and $\{x, y\}$ is an edge in $\mathcal{G}$ if x can be obtained from y by a minimal change. A method of generation $\mathcal{V}$ by minimal changes is a method to traverse all vertices 'moving' only on edges i. e. is equivalent to a Hamilton path in $\mathcal{G}$. Graphs of minimal changes are helpful both to construct algorithms of minimal changes for some combinatorial objects and to prove, that such algorithms do not exist for some other combinatorial objects.

If $\mathcal{G}$ is bipartite, then the most typical tool for checking the possibility of construction of a minimal change generation algorithm is the *parity difference*. A bipartite graph is a graph whose vertex set $\mathcal{V}$ can be split into two disjoint subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that there are no edges between vertices belonging to the same subset. Graphs of minimal changes corresponding to generation problems considered by many authors (which includes those cited at the end of this paper) are bipartite. If there exists a Hamilton path in a such a bipartite graph of minimal changes, then the difference between the number of vertices in $\mathcal{V}_1$ and $\mathcal{V}_2$ may not be greater than 1. This difference is called the parity difference $d(\mathcal{G})$ of $\mathcal{G}$. For convenience in this paper we also denote it by $d(\mathcal{V})$.

An *involution* on $\mathcal{V}$ is a bijection $\phi : \mathcal{V} \to \mathcal{V}$, such that $\phi = \phi^{-1}$. A *fixpoint* of an involution ϕ is $x \in \mathcal{V}$, such that $\phi(x) = x$. Involution ϕ can make it easy to compute parity difference, if for any $x \neq \phi(x)$, x and $\phi(x)$ have different parities, and parity difference for the set of all fixpoints of ϕ can be easily determined. In such case those two parity differences are equal, because all $x \in \mathcal{V}$ not being fixpoints do not influence parity difference. Enumeration various numbers related to combinatorial objects by involutions is considered in [11] and [9]. In this paper we show involution solutions of two problems concerning parity difference.

2 Parity difference for linear extensions

In this paper we consider combinatorial objects being strings of elements of given set $\mathcal{A}$ with some constraints (if no constraints are imposed they are simply permutations). Minimal change is transposition of two adjacent elements of a string. Graph of minimal changes of such combinatorial objects is bipartite as it is a subgraph of the graph of all permutations of $\mathcal{A}$ in which odd and even permutations form the bipartition.

One of the ways of imposing some constraints on permutations is to define a partial order $<$ on $\mathcal{A}$. The strings to be generated are linear extensions of poset $P = (A, <)$. Let $\mathcal{L}(P)$ denote the set of linear extensions of P. Ruskey [8] posed a conjecture that linear extensions of a poset can be generated by (adjacent) transpositions if $|d(\mathcal{L}(P))| \leq 1$. A partial solution for series-parallel posets can be found in [6]. We prove that enumeration of $d\left(\mathcal{L}(P)\right)$ is in general equally difficult as enumeration of the number of linear extensions $|\mathcal{L}(P)|$, i. e. is a $\#P$-complete problem.

THEOREM 1 *Computing parity difference for linear extensions of a poset is a $\#P$-complete problem.*

Proof: It is not hard to construct two polynomial time non–deterministic algorithms whose numbers of accepting paths are equal to the numbers of linear extensions being either odd or even permutations respectively. So parity difference as the difference of these two numbers is in $\#P$.

In order to carry out the proof in the other direction we assign to an arbitrary poset P another poset Q. Each $p \in P$ corresponds to two vertices $s_p, t_p \in Q$. We have two kinds of relations in Q. The first is $s_p < t_p$ for each $p \in P$. The second is given by poset P. If $p, r \in P, p < r$, then $t_p < s_r$.

We prove that $d\left(\mathcal{L}(Q)\right) = |\mathcal{L}(P)|$. First introduce an involution ϕ on linear extensions of Q. We have given a linear extension $q_1 q_2 q_3 \cdots q_{2n}$ of Q. Let i be the smallest index such that q_{2i-1} and q_{2i} are incomparable in Q. The involution ϕ swaps q_{2i-1} and q_{2i}. If there is no such i, then ϕ does nothing.

The involution ϕ changes the parity of extensions of Q that are not its fixpoints. All the fixpoints of ϕ have the form $s_{\sigma(1)} t_{\sigma(1)} s_{\sigma(2)} t_{\sigma(2)} \cdots s_{\sigma(n)} t_{\sigma(n)}$ and the same parity, where σ is a permutation of $\{1, 2, \ldots, n\}$. Thus $d\left(\mathcal{L}(Q)\right)$ is equal to the number of these fixpoints. On the other hand, each such fixpoint corresponds to a linear extension $p_{\sigma(1)} p_{\sigma(2)} p_{\sigma(3)} \cdots p_{\sigma(n)}$ of P. So $d\left(\mathcal{L}(Q)\right) = |\mathcal{L}(P)|$.

If we had a polynomial algorithm for counting parity difference we also would be able to compute the number of linear extensions of an arbitrary poset in polynomial time. But counting linear extensions was proved to be $\#P$-complete in [2].□

3 Parity difference for trees

We denote $\vec{n} = (n_0, n_1, \ldots, n_k)$ and $n = n_0 + n_1 + \cdots + n_k$. Let $\vec{e}_a = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where this single 1 is in the $(a+1)$-st position in this string. We introduce $\vec{e}_a$ in order to denote $(n_0, n_1, \ldots, n_a \pm 1, \ldots, n_k)$ as $\vec{n} \pm \vec{e}_a$. Also define

$$\left\lfloor \frac{\vec{n}}{2} \right\rfloor \stackrel{\text{def}}{=} \left(\left\lfloor \frac{n_0}{2} \right\rfloor, \left\lfloor \frac{n_0}{2} \right\rfloor, \ldots, \left\lfloor \frac{n_k}{2} \right\rfloor \right).$$

Let us define the set $\mathcal{C}(\vec{n})$ of combinatorial objects being the strings $A = a_1 a_2 a_3 \ldots a_n$ consisting of n_0 0's, n_1 1's, $\ldots$, n_k k's. These strings are called *multiset permutations.* If we denote

$$C(\vec{n}) \stackrel{\text{def}}{=} \frac{n!}{n_0!n_1!\cdots n_k!},$$

then the number of elements in $\mathcal{C}(\vec{n})$ is equal $C(\vec{n})$. In this paper we consider the set $\mathcal{T}(\vec{n})$ as being the subset of $\mathcal{C}(\vec{n})$ consisting of strings A fulfilling the condition

$$a_1 + a_2 + \cdots + a_i \leq i \text{ for all } i \in \{1, \ldots, n\}.$$

The "minimal change" in both cases is a transposition of two adjacent elements of the string.

If we assign a label to each character in one of strings in $\mathcal{C}(\vec{n})$ or $\mathcal{T}(\vec{n})$, then the strings are permutations of these labels fulfilling some constraints. The graph of all permutations is bipartite because odd and even permutations form partition sets of its vertices. Thus, the graph of minimal changes $\mathcal{G}$, both for $\mathcal{C}(\vec{n})$ and for $\mathcal{T}(\vec{n})$, is bipartite as it is a subgraph of the graph of all permutations. The parity difference for $\mathcal{C}(\vec{n})$ was computed in [5] [7] and is equal to

$$d(\mathcal{C}(\vec{n})) = \begin{cases} C\left(\left\lfloor \dfrac{\vec{n}}{2} \right\rfloor\right) & \text{if } \vec{n} \text{ contains no more than one odd entry,} \\ \\ 0 & \text{otherwise.} \end{cases}$$

It was also proven [10] (see also [7]), that whenever $d(\mathcal{C}(\vec{n}))$ is 0 or ± 1 the elements of $\mathcal{C}(\vec{n})$ can be generated by adjacent transpositions. The aim of this paper is presentation of a method of finding parity difference $d(\mathcal{T}(\vec{n}))$ alternative to that described by Ko and Ruskey [5]. This method is similar to method described in [7] for $d(\mathcal{C}(\vec{n}))$ and gives better insight into the structure of the result.

For $A \in \mathcal{T}(\vec{n})$ and an $i, 1 \leq i \leq n$ we define the constraint function

$$h(A, i) = i - (a_1 + \cdots + a_i).$$

The set $\mathcal{T}(\vec{n})$ is the set of multiset permutations A fulfilling the condition $h(A, i) \geq 0$ for $i = 1, 2, \ldots, n$. We say that A has a *threshold* after a_i if $h(A, i) = 0$. We notice that $h(A, n)$ is the same for each $A \in \mathcal{T}(\vec{n})$ and is equal to

$$h(\vec{n}) = \sum_{a=0}^{k} (1 - a) n_a.$$

The strings $0A$, where $A \in \mathcal{T}(\vec{n})$ are equivalent to ordered forests having $n_0 + 1$ leaves, n_1 vertices having 1 child, n_2 vertices having 2 children, ..., as strings of numbers of children listed in postorder [14]. The value $h(\vec{n}) + 1$ is the number of trees in such a forest. Let $T(\vec{n})$ denote the number of elements of $\mathcal{T}(\vec{n})$.

Theorem 2 (see [14]) *If $n \neq 0$, then*

$$T(\vec{n}) = \frac{C(\vec{n})}{n_0+1} \cdot (h(\vec{n})+1) = \frac{C(\vec{n}+\vec{e}_0)}{n+1} \cdot h(\vec{n}+\vec{e}_0). \square$$

Now we introduce some definitions neccessary to formulate the main theorem of this section. Let S be a subset of set of natural numbers N. By $\mathcal{T}_S(\vec{n})$ we denote the subset of strings in $\mathcal{T}(\vec{n})$ that have no thresholds after a_i's belonging to S. Such strings represent forests with no internal nodes belonging to S in the leftmost branch of their first trees. The number of strings in $\mathcal{T}_S(\vec{n})$ we denote by $T_S(\vec{n})$. Set of even numbers (with zero) is denoted by P.

Multiset permutations that are obtained by even number of adjacent transpositions from $A_0 = 00\cdots01\cdots12\cdots kk$ we call even. Let $d(\mathcal{T}(\vec{n})) =$ (# even permutations − # odd permutations). Let $\Theta(\vec{n})$ denote the number of odd n_a with a – odd, and $\mathrm{E}(\vec{n})$ denote the number of odd n_a with a – even; $a \neq 0$. Denote $n_S = \sum_{a\in S} n_a$ and $h_S(\vec{n}) = \sum_{a\in S}(1-a)n_a$. Notice that $n = n_N$ and $h(\vec{n}) = h_N(\vec{n})$.

Now we remind the result of Ko and Ruskey, whose alternative proof we find. Let $\vec{m} = \left\lfloor \frac{\vec{n}}{2} \right\rfloor$.

Theorem 3 (see [5])

(A) *If $n = n_1$, then $d(\mathcal{T}(\vec{n})) = 1$.*

(B) *If $\Theta(\vec{n}) \leq 1, n_0$ is odd, $\mathrm{E}(\vec{n}) = 1$, then*

$$d(\mathcal{T}(\vec{n})) = \rho C\left(\vec{m}\right) \frac{m+1}{m_0+1} \left[\frac{h_P(\vec{m})}{m_P+1} + \frac{h_{N\setminus P}(\vec{m})}{m_P+2}\right].$$

where $\rho = 1$ if $\Theta(\vec{n}) = 0$, and $\rho = (-1)^{n_{\{i:i<j\}}}$ where j is the only index such that n_j is odd.

(C) *If $\Theta(\vec{n}) \leq 1, n_0$ is even, $\mathrm{E}(\vec{n}) = 0$, then*

$$d(\mathcal{T}(\vec{n})) = C\left(\vec{m}\right) \left[\frac{h_P(\vec{m})}{m_P} + \frac{h_{N\setminus P}(\vec{m})}{m_P+1}\right].$$

(D) *If $\Theta(\vec{n}) = 0, n_0$ is odd, $\mathrm{E}(\vec{n}) = 0$, then*

$$d(\mathcal{T}(\vec{n})) = T\left(\vec{m}\right).$$

(E) *In all other cases $d(\mathcal{T}(\vec{n})) = 0.$* $\square$

Now we begin to compute $d(\mathcal{T}(\vec{n}))$. This is done by reduction of pairs of strings that have opposite parities. Let A be a string in $\mathcal{T}(\vec{n})$. From now on, we will consider $A = a_1a_2a_3\ldots a_n$ as a composition of a sequence of ordered pairs : $a_1a_2, a_3a_4, a_5a_6\ldots$ and a single element a_n if n is odd. We say there is a threshold after some pair if it appears after the second element of this pair. We define involution $\psi : \mathcal{T}(\vec{n}) \to \mathcal{T}(\vec{n})$ in the following way:

ψ_1 If at least one pair of which A is composed can be transposed and the resulting string belongs to $\mathcal{T}(\vec{n})$, then $\psi(A)$ is equal to such a string produced by transposing the first such pair.

ψ_2 If the previous condition is not fulfilled we consider all pairs in the sequence having the form $0a$; $a \neq 0$. We pair up such pairs: the first with the second, the third with the fourth ... (the last pair can be unpaired). If such a pair of pairs exists $(0a, 0b)$, with $a \neq b$, then we perform the following sequence of operations. First replace $0a$ with 00 and $0b$ with $0a$. Then replace with $0b$ the first pair 00, for which the resulting string belongs to $\mathcal{T}(\vec{n})$.

ψ_3 If no of previous operations can be performed, we consider the first pair having the form $0a$ ($a \neq 0$) or aa with a-even. If such a pair is of the form aa, then we produce $\psi(A)$ replacing this pair by $0a$ and substituting by $0a$ the first pair 00 for which the resulting string belongs to $\mathcal{T}(\vec{n})$. If such a pair is of the form $0a$ and the second pair $0a$ exists, then we produce $\psi(A)$ by replacement of the first pair $0a$ by 00 and the second by aa.

ψ_4 If neither of the previous cases apply, then $\psi(A) = A$.

We can also view $\psi_{1,2,3,4}$ as four involutions – each defined on a separate domain described in its definition. Now we prove the following lemma stating the most important properties of ψ.

LEMMA 1 *Function ψ has the following properties:*

1. *Function ψ is an involution.*
2. *If A is not a fixpoint of ψ, then A and $\psi(A)$ have opposite parities.*
3. *Fixpoints of ψ are composed of three kinds of pairs:*
   - *aa for a-odd.*
   - *at most one pair $0b$ (b-even) and there is a threshold after this pair.*
   - *cc for c-even — if this pair is not preceded by a pair $0b$, then there is no threshold after this pair.*

Before the proof of Lemma 1 we formulate two additional technical lemmas helpful in this proof.

LEMMA 2 *Let us have a string A in $\mathcal{T}(\vec{n}+\vec{e}_0-\vec{e}_c)$ composed of pairs aa and $0b$ for b-even. Let $h(\vec{n})>c$. There is exactly one way of substituting a pair 00 by $0c$ such that the resulting sequence belongs to $\mathcal{T}(\vec{n})$ and there is a threshold after this $0c$.*

Proof: This only pair 00 is the first pair, after which the function $h(A,*)$ is greater than c.□

The second lemma we give without a proof, that would consist in constructing a sequence of adjacent transpositions transforming one string into another.

LEMMA 3 *Let us have a string A in $\mathcal{T}(\vec{n})$ composed of pairs aa and $0b$ for b-even. We consider two pairs $0a$ and $0b$ such that there is no pair $0c$ between them ($a,b,c\neq 0$). If we replace $0a$ by 00 and $0b$ by ab, then the resulting sequence differs from A by an odd number of adjacent transpositions. In other words they have opposite parities in graph of minimal changes.*

With above two lemmas we can prove Lemma 1
Proof of Lemma 1: Obviously conditions 1. and 2. of the lemma are fulfilled for strings for which ψ_1 and ψ_4 are performed. There are two kinds of pairs of which A is composed, which cannot be transposed in ψ_1 if the resulting string is to belong to $\mathcal{T}(\vec{n})$. They are pairs of the form aa and $0b$ (b-even), where we have a threshold after $0b$. So strings that are not subject to ψ_1 consist of such pairs. Now we consider all such pairs of the form $0a$ in the string. We pair them up, and if we have such a pair of pairs $(0a,0b)$, that $a\neq b$, we can apply ψ_2 to the string. Operation ψ_2 is well defined and is an involution because of Lemma 2. Function ψ_2 can be viewed as a superposition of three operations (denotations as in the definition):

1. exchanging $0a$ by 00 and $0b$ by ab,
2. adjacent transposition of the pair ab,
3. exchanging ba by $0a$ and exchanging 00 by $0b$ for the first such a pair 00 for which the resulting string belongs to $\mathcal{T}(\vec{n})$.

All three operations require an odd number of adjacent transpositions (the first and the last because of Lemma 3), so ψ_2 requires an odd number of adjacent transpositions. All strings for which we cannot apply both ψ_1 and ψ_2 consist of pairs aa or $0b$, where there is a threshold after $0b$ and $2i$-th and $(2i+1)$-st pair of the form $0b$ are identical for all integers i. Now we consider ψ_3. It is defined separately for strings in which there exists a pair aa; a-even, that precedes all pairs $0b$ and for strings in which such a pair does not exist. Both functions defined this way and constituting ψ_3 altogether are well defined and mutually reverse due to Lemma 2, so ψ_3 is an involution. Because of Lemma 3, involution ψ_3 changes parity of the string. It is not hard to see, that what is left after ψ_1, ψ_2 and ψ_3 are fixpoints described in the lemma.□

We also need the following simple lemma for our computations:

Lemma 4

1. *The number of strings in $\mathcal{T}(\vec{n})$ consisting of pairs aa, after which there is no threshold if a is even is equal*

$$T_P\left(\left\lfloor\frac{\vec{n}}{2}\right\rfloor\right).$$

2. *The number of strings in $\mathcal{T}(\vec{n})$ consisting of one pair $0b$ with a threshold after it, and pairs aa after which there is no threshold, if a is even and such a pair precedes $0b$ is equal*

$$T_P\left(\left\lfloor\frac{\vec{n}}{2}\right\rfloor + e_0\right).$$

Proof: Part 1. of the lemma can be easily obtained by substitution for each pair aa a single symbol a. Part 2. is obtained from 1. by establishing a bijection between these two kinds of strings. This bijection exchanges the pair $0b$ for the pair 00 in strings from part 2. The reverse operation is well defined because of Lemma 2. □

Now we enumerate $d(\mathcal{T}(\vec{n}))$. We consider the set $\mathcal{F}$ of the fixpoints of ψ to be a sum of two disjoint subsets $\mathcal{F}_i$, where $i = 0, 1$ denotes the number of pairs $0a$ in fixpoints of which they consist. To enumerate $d(\mathcal{F}_i)$ we use lemmas 1 and 4. We label the cases following denotations from [5]. First we write down the cases for $d(\mathcal{F}_0)$.

(A0)(D0) If $n = 0, 1$, then $d(\mathcal{F}_0) = 1$.

(A1)(C)(D1)(E1) If $\Theta(\vec{n}) + \mathrm{E}(\vec{n}) + (n_0 \bmod 2) \leq 1$, then strings in $\mathcal{F}_0$ have the form $ccdd\cdots zz(b)$ and the same positive parity. So

$$d(\mathcal{F}_0) = T_P\left(\left\lfloor\frac{\vec{n}}{2}\right\rfloor\right).$$

(B)(E2)(E3) In all other cases $\mathcal{F}_0 = \emptyset$.

Then we enumerate $d(\mathcal{F}_1)$:

(B) If $\Theta(\vec{n}) \le 1, n_0$ is odd, $\mathrm{E}(\vec{n}) = 1$, then strings in $\mathcal{F}_1$ have the form $ccdd \cdots 0a \cdots zz(b)$ and the same parity (b is odd). If $b > a$ or there is no such b, then this parity is positive. If $b < a$ it is negative.

$$|d(\mathcal{F}_1)| = T_P\left(\left\lfloor \frac{\vec{n}}{2} \right\rfloor + \vec{e}_0\right).$$

(E1) If $\Theta(\vec{n}) = 0, n_0$ is even, $\mathrm{E}(\vec{n}) = 1$, then strings in $\mathcal{F}_1$ have the form $ccdd \cdots 0a \cdots zz0$ and the same negative parity. So,

$$d(\mathcal{F}_1) = -T_P\left(\left\lfloor \frac{\vec{n}}{2} \right\rfloor\right).$$

(D) If $\Theta(\vec{n}) = 0, n_0$ is odd, $\mathrm{E}(\vec{n}) = 0$, then strings in $\mathcal{F}_1$ have the form $ccdd \cdots 0a \cdots zza$ for all even a and the same positive parity. So,

$$d(\mathcal{F}_1) = \sum_{0 \neq a \text{ even}} T_P\left(\left\lfloor \frac{\vec{n}}{2} \right\rfloor + \vec{e}_0 - \vec{e}_a\right).$$

(E2) If $\Theta(\vec{n}) = 0, n_0$ is odd, $\mathrm{E}(\vec{n}) = 2$, then strings in $\mathcal{F}_1$ have either the form $ccdd \cdots 0a \cdots zzb$ or $ccdd \cdots 0b \cdots zza$ (n_a and n_b are odd). These two kinds of strings have opposite parities so,

$$d(\mathcal{F}_1) = T_P\left(\left\lfloor \frac{\vec{n}}{2} \right\rfloor + \vec{e}_0\right) - T_P\left(\left\lfloor \frac{\vec{n}}{2} \right\rfloor + \vec{e}_0\right) = 0.$$

(A)(C)(E3) In all other cases $\mathcal{F}_1 = \emptyset$.

Now we can enumerate $d(\mathcal{T}(\vec{n})) = d(\mathcal{F}_0) + d(\mathcal{F}_1)$:

(B) If $\Theta(\vec{n}) \le 1, n_0$ is odd, $\mathrm{E}(\vec{n}) = 1$, then

$$d(\mathcal{T}(\vec{n})) = \pm T_P\left(\left\lfloor \frac{\vec{n}}{2} \right\rfloor + \vec{e}_0\right).$$

The sign of $d(\mathcal{T}(\vec{n}))$ depends on n_a (a – even) and n_b (b – odd) that are odd. If $a > b$, then this sign is negative. Otherwise (also if there is no such b) it is positive.

(A)(C) If $\Theta(\vec{n}) \le 1, n_0$ is even, $\mathrm{E}(\vec{n}) = 0$, then

$$d(\mathcal{T}(\vec{n})) = T_P\left(\left\lfloor \frac{\vec{n}}{2} \right\rfloor\right).$$

(D) If $\Theta(\vec{n}) = 0, n_0$ is odd, $\mathrm{E}(\vec{n}) = 0$, then

$$d(\mathcal{T}(\vec{n})) = \sum_{a \text{ even}} T_P\left(\left\lfloor \frac{\vec{n}}{2} \right\rfloor + \vec{e}_0 - \vec{e}_a\right).$$

(E1) If $\Theta(\vec{n}) = 0, n_0$ is even, $\mathrm{E}(\vec{n}) = 1$, then

$$d(\mathcal{T}(\vec{n})) = T_P\left(\left\lfloor \frac{\vec{n}}{2} \right\rfloor\right) - T_P\left(\left\lfloor \frac{\vec{n}}{2} \right\rfloor\right) = 0.$$

(E2)(E3) In all other cases both $d(\mathcal{F}_0)$ and $d(\mathcal{F}_1)$ are equal to zero.

4 Enumeration of the number of trees with constraints

We now prove two results that give us explicit formulas for $d(\mathcal{T}(\vec{n}))$. They are some generalizations of the formulas that can be derived from the main theorem in [5]. First we introduce some denotations. Let N be the set of natural numbers. Notice that

$$T_{\{0\}}(\vec{n}) = T(\vec{n})$$

and

$$T_N(\vec{n}) = T(\vec{n} - \vec{e}_0). \tag{1}$$

The second equality holds, because each $0A \in \mathcal{T}_N(\vec{n})$ corresponds to $A \in \mathcal{T}(\vec{n} - \vec{e}_0)$.

Lemma 5 *For all $S \subseteq N$, where $0 \in S$,*

$$T(\vec{n} - \vec{e}_0) = \sum_{a \in S} T_S(\vec{n} - \vec{e}_a).$$

Proof: The number of strings in $\mathcal{T}(\vec{n} - \vec{e}_0)$ that contain no $a_i \in S$ after which there is a threshold, is equal to $T_S(\vec{n} - \vec{e}_0)$. On the other hand there is one-to-one correspondence between strings in $\mathcal{T}(\vec{n} - \vec{e}_0)$ having the smallest i for which there is a threshold after $a_i \in S$ such that $a_i = a$, and $\mathcal{T}_S(\vec{n} - \vec{e}_a)$ given by exchanging a_i by 0. Thus the number of such strings is equal $T_S(\vec{n} - \vec{e}_a)$. Summing up these formulas gives us a formula for $T(\vec{n} - \vec{e}_0)$.□

If $n = n_1$, then obviously $T_S(\vec{n}) = 1$. Now we find $T_S(\vec{n})$ for other $\vec{n}$.

Theorem 4 *If $n > n_1$, then*

$$T_S(\vec{n}) = C(\vec{n}) \cdot \left(\frac{h_S(\vec{n})}{n_S} + \frac{h_{N\setminus S}(\vec{n})}{n_S + 1}\right).$$

Proof: We proceed by induction on $n_S^* = n_{N\setminus S}$. If $n_S^* = 0$, then by Theorem 2 and (1) we have

$$T_S(\vec{n}) = \frac{C(\vec{n})}{n}h(\vec{n}) = C(\vec{n}) \cdot \left(\frac{h_S(\vec{n})}{n_S} + \frac{h_{N\setminus S}(\vec{n})}{n_S+1}\right).$$

If $n_S^* > 0$, then we have a one-to-one correspondence between strings in $\mathcal{T}_S(\vec{n})$ that have $a_n = a$ and $\mathcal{T}_S(\vec{n} - \vec{e}_a)$ given by deletion of a_n. So

$$T_S(\vec{n}) = \sum_{a \in N} T_S(\vec{n} - \vec{e}_a).$$

Because of Lemma 5 we have

$$T_S(\vec{n}) = T(\vec{n} - \vec{e}_0) + \sum_{a \notin S} T_S(\vec{n} - \vec{e}_a).$$

By inductive hypothesis and Theorem 2:

$$T_S(\vec{n}) = \frac{C(\vec{n})}{n}h(\vec{n}) + \sum_{a \notin S} C(\vec{n} - \vec{e}_a)\left(\frac{h_S(\vec{n} - \vec{e}_a)}{n_S} + \frac{h_{N\setminus S}(\vec{n} - \vec{e}_a)}{n_S+1}\right)$$

$$= \frac{C(\vec{n})}{n}h(\vec{n}) + \sum_{a \notin S} C(\vec{n})\frac{n_a}{n}\left(\frac{h_S(\vec{n})}{n_S} + \frac{h_{N\setminus S}(\vec{n}) - (1-a)}{n_S+1}\right)$$

$$= \frac{C(\vec{n})}{n}\left(h(\vec{n}) + \frac{n_{N\setminus S}h_S(\vec{n})}{n_S} + \frac{n_{N\setminus S}h_{N\setminus S}(\vec{n})}{n_S+1} - \frac{h_{N\setminus S}(\vec{n})}{n_S+1}\right)$$

$$= C(\vec{n})\left(\frac{h_S(\vec{n})}{n_S} + \frac{h_{N\setminus S}(\vec{n})}{n_S+1}\right). \square$$

Using Lemma 5 we can formulate the results about parity difference for forests in the following way (cases labelled as in [5]):

THEOREM 5

(B) *If* $\Theta(\vec{n}) \le 1, n_0$ *is odd*, $\mathrm{E}(\vec{n}) = 1$, *then*

$$d(\mathcal{T}(\vec{n})) = \pm T_P\left(\left\lfloor\frac{\vec{n}}{2}\right\rfloor + \vec{e}_0\right).$$

The sign of $d(\mathcal{T}(\vec{n}))$ *depends on* n_a *(a – even) and* n_b *(b – odd) that are odd. If* $a > b$*, then this sign is negative. Otherwise (also if there is no such* b*) it is positive.*

(A+C) *If* $\Theta(\vec{n}) \leq 1, n_0$ *is even*, $\mathrm{E}(\vec{n}) = 0$, *then*

$$d(\mathcal{T}(\vec{n})) = T_P\left(\left\lfloor \frac{\vec{n}}{2} \right\rfloor\right).$$

(D) *If* $\Theta(\vec{n}) = 0, n_0$ *is odd*, $\mathrm{E}(\vec{n}) = 0$, *then*

$$d(\mathcal{T}(\vec{n})) = T\left(\left\lfloor \frac{\vec{n}}{2} \right\rfloor\right).$$

(E) *In all other cases* $d(\mathcal{T}(\vec{n})) = 0.\square$

Now we formulate the conditions describing when $T_S(\vec{n})$ is equal zero or one. The conditions when $|d(\mathcal{T}(\vec{n})| = 0, 1$ are gathered in the next corollaries, that follow from this last result and Theorem 5. They describe the cases when we can hope to construct a minimal change generation algorithm for $\mathcal{T}(\vec{n})$ and differ slightly from the results in [5]. To proceed further we formulate the following simple lemma without a proof:

LEMMA 6 *If* $T_S(\vec{n}) > 0$, *then a string* A *having all zeros at the beginning and all symbols not belonging to* S *at the end, belongs to* $\mathcal{T}_S(\vec{n})$.

The conditions describing when $T_S(\vec{n}) \in \{0, 1\}$ are gathered in the next lemma.

LEMMA 7 *Let* $h(\vec{n}) \geq 0$.

1. $T_S(\vec{n}) = 0$ *if and only if* $n = n_{S\cup\{1\}}$ *and* $h(\vec{n}) = 0$.
2. $T_S(\vec{n}) = 1$ *if and only if one of the following conditions holds*
   - *(a)* $n = n_1$
   - *(b)* $a \in S, 2 \notin S, a = 1$ *or* $n_a = 1, n_2 = 1, n_0 = a, n = n_0 + n_a + n_2$
   - *(c)* $b \notin S, b = 1$ *or* $n_b = 1, n_0 = b - 1, n = n_0 + n_b$
   - *(d)* $a \in S, a = 1$ *or* $n_a = 1, n_0 = a, n = n_0 + n_a$
   - *(e)* $n = n_0$

Proof: First we prove part 1. We notice, that strings from Lemma 6 cannot belong to $\mathcal{T}_S(\vec{n})$. It is not hard to see, that all $a \notin S$ have to be ones and $h(\vec{n}) = 0$. Now we consider part 2. There should be only one string of the type described in Lemma 6. This string has the form $00\ldots0a\ldots ab\ldots b$, where $a \in S \setminus \{0\}$ and $b \notin S$ (the number of symbols 0, a or b can be zero). We have to find the conditions when there are no other strings in $\mathcal{T}_S(\vec{n})$. If $b = 1$, then there are no other symbols (case (a)), because otherwise transposing leftmost 1 in the string with its left neighbor we obtain a string belonging to $\mathcal{T}_S(\vec{n})$. If $b > 1$ and there are some symbols a, then transposing the leftmost a with its neighbor we do not produce a string in $\mathcal{T}_S(\vec{n})$ only if $b = 2$, $h(\vec{n}) = 0$ and also $a = 1$ or there is only one symbol a (case (b)). If $b > 1$ and there are no symbols a, then transposing the leftmost b with its neighbor does not give a string in $\mathcal{T}_S(\vec{n})$ only if $h(\vec{n}) = 0$ and there is only one symbol b (case (c)). If there are no symbols b, then transposing the leftmost a with its neighbor we do not produce a string in $\mathcal{T}_S(\vec{n})$ only if $h(\vec{n}) = 1$ and also $a = 1$ or there is only one symbol a (as in (d)). There remains the case (e) where we have no symbols other than zero. □

COROLLARY 1 *$d(\mathcal{T}(\vec{n})) = 0$ if and only if*

- *we have one of the 'other' cases described in Theorem 5.*
- *$n = n_P + n_1$ and $\mathrm{E}(\vec{n}) = 0$ and $h(\vec{n}) = 0$.*□

COROLLARY 2 *$d(\mathcal{T}(\vec{n})) = \pm 1$ if and only if*

- $n = n_1$,
- $n_1 \in P, n_0 = 1, n = n_0 + n_1$,
- $n_0 \notin P, a \in P, n_a = 1, b \notin P, n_b \leq 1, n = n_0 + n_a + n_b$,
- $n_0 \in P, b \notin P, n_b \leq 1, n = n_0 + n_b$,
- $n = n_0$,
- $b \notin P, n_0 = 2(b-1), n_b = 2, n_1 \leq 1, n = n_0 + n_1 + n_b$,
- $a > 0, n_0 = 2a - 1, n_a = 2, n = n_0 + n_a$,
- $n_0 = 3, n_2 = 3, n_1 \leq 1, n = n_0 + n_1 + n_2$,
- $a \in P, n_0 = 2a, n_a = 2, n_1 + n_3 \leq 1, n = n_0 + n_1 + n_3 + n_a$.□

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