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HAMILTON PATHS IN GRAPHS OF LINEAR EXTENSIONS FOR UNIONS OF POSETS*

GRZEGORZ STACHOWIAK†

Abstract. This paper proves that if a poset Q has an even number of linear extensions and these extensions can be generated by adjacent transpositions, then linear extensions of union of poset Q and an arbitrary poset P can also be generated by adjacent transpositions. This result is then applied to posets P and Q, which are sums of disjoint chains.

Key words. linear extension, Hamilton path, Gray code

AMS(MOS) subject classification. 05C45

1. Introduction. Let P be a partially ordered set (poset) and $\mathscr{L}(P)$ denote the set of all its linear extensions. For two posets P and Q, we define their *parallel* $P|Q$ and *series* PQ compositions. The poset $P|Q$ is the union of P and Q, and the poset PQ is constructed from $P|Q$ by adding the relations that state that each element of P is greater than each element of Q. Let i^n denote a chain of n elements labeled with i; i^0 denotes the empty chain.

A linear extension $L_2 \in \mathscr{L}(P)$ can be obtained from another linear extension $L_1 \in \mathscr{L}(P)$ by an *elementary transformation* if L_1 and L_2 differ by the order of exactly two adjacent elements. We define the graph $GE(P)$ of all linear extensions of P with the elements of $\mathscr{L}(P)$ as its vertices and two vertices form an edge in $GE(P)$ if the corresponding linear extensions differ by a single interchange of adjacent elements. Note that a Hamilton path in $GE(P)$ implies the existence of an algorithm that generates all linear extensions of P by a sequence of elementary transformations. In general, we may consider an arbitrary family of combinatorial objects and ask for such an algorithm, which is called the *Gray code* of these objects. For two posets P and Q on the same ground set, the graph of $GE(P, Q)$ is defined analogously. The vertex set of $GE(P|Q)$ is $\mathscr{L}(P) \cup \mathscr{L}(Q)$, and two vertices are adjacent if they differ by an elementary transformation (i.e., by transposition of two adjacent elements in linear extensions).

In some cases, the problem of generating some combinatorial objects can be transformed to generating linear extensions of a properly defined poset. A motivation for this paper is the following problem: how do we generate all strings containing precisely $n(1) - 1$'s, $n(2) - 2$'s, $\cdots$, $n(k) - k$'s? The existence of Gray code for such strings is equivalent to the existence of a Hamilton path in the graph $H = GE(1^{n(1)}|2^{n(2)}| \cdots |k^{n(k)})$. This problem was considered by Ko and Ruskey [6].

It is well known that the graph $G = GE(P)$ is bipartite and connected for any poset P. Let $V_1 \cup V_2$ be a bipartition of $V(G)$ and $d(G) = \|V_1| - |V_2\|$. There is no Hamilton path in $GE(P)$ if $d(G)$ exceeds 1. Ko and Ruskey gave the formula for $d(H)$, and they posed the following conjecture.

CONJECTURE. *Graph H has a Hamilton path if and only if $d(H)$ is 0 or 1.*

This conjecture has been proved by Ruskey [8] for transpositions of not necessarily adjacent elements as well as for adjacent transpositions in some special cases. We show it for all graphs H.

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† Institute of Computer Science, University of Wrocław, Przesmyckiego 20, 51-151 Wrocław, Poland.
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It was shown in [6] that $d(H)$ is 1 if $k = 1$ or $k = 2$ and either $n(1)$ or $n(2)$ is equal to 1 and the remaining one is even. It is not difficult to prove the conjecture in this case. The value $d(H)$ is zero if and only if for some i and j ($i \neq j$), $n(i)$ and $n(j)$ are both odd; without loss of generality, we can assume that $i = 1$ and $j = 2$. It is well known that $GE(1^{n(1)}|2^{n(2)})$ has a Hamilton path for $n(1)$ and $n(2)$ odd; see [2], [3], and [7]. For k greater than 2, the conjecture follows from our main result.

THEOREM. *If for a poset Q the graph $GE(Q)$ has an even number of vertices and contains a Hamilton path, then the graph $GE(Q|P)$ has a Hamilton path for every poset P.*

If $Q = 1^{n(1)}|2^{n(2)}$, where $n(1)$ and $n(2)$ are odd, and $P = 3^{n(3)}| \cdots |k^{n(k)}$, then the conditions of the theorem are fulfilled (see [2]); therefore, the conjecture is true.

In what follows, we will make frequent use of the operation of gluing two edges. Edges $e_1 = [a, b]$ and $e_2 = [c, d]$ of a graph G are *parallel* if G contains edges $f_1 = [a, c]$ and $f_2 = [b, d]$. Let us assume that there exist two vertex-disjoint cycles C_1 and C_2 in G such that C_1 contains e_1 and C_2 contains e_2. The *gluing* of edges e_1 and e_2 consists of exchanging e_1 and e_2 for f_1 and f_2. This operation produces one cycle containing all the vertices of cycles C_1 and C_2.

2. The theorem for Q consisting of two isolated elements. In this section we prove the theorem for $Q = 1|2$. An edge in $GE(Q|l^p)$ is called an *i-edge* if both its end vertices correspond to linear extensions of $Q|l^p$ in which the ith and $(i+1)$th elements of l^p are adjacent. A set consisting of i-edges e_i ($1 \leqq i < p$) such that $e_i \neq e_j$ for $i \neq j$ is called a *full set of i-edges*. We first formulate an obvious lemma.

LEMMA 1. *If a vertex $x \in G$ of degree 2 belongs to a cycle in G, then the cycle contains both of the edges incident to x.*

Now we prove some properties of graph $GE((1|2)|l^p)$ that will be useful later.

LEMMA 2. *The graph $G = GE((1|2)|l^p)$ has the following properties:*

1. *If p is odd, then G contains a Hamilton cycle containing a full set of i-edges.*

2. *If p is even, then G contains two cycles. Each of these cycles contains all but two vertices of G. The omitted vertices are either*

2.1. $12l^p$ *and* $21l^p$ (*see Fig.* 1) *or*

2.2. l^p12 *and* l^p21.

Moreover, if $p \neq 2$ then both cycles have the same full set of i-edges.

Proof. The graph G has $p+1$ vertex-disjoint paths d_i ($0 \leqq i \leqq p$) of the form

$$d_i = [1l^{p-i}2l^i, l1l^{p-i-1}2l^i, \cdots, l^{p-i}12l^i, l^{p-i}21l^i, \cdots, 2l^{p-i}1l^i],$$

and these paths cover all the vertices of G. Let a_i denote the last edge of d_i, $0 \leqq i \leqq p$. Connecting the ends of d_0 and d_1, d_2 and d_3, $\cdots$ we obtain $\lfloor (p+1)/2 \rfloor$ disjoint cycles in G and if p is even, there exists one additional path d_p, which is the edge $[12l^p, 21l^p]$. Then, by gluing a_1 and a_2, a_3 and a_4, and so on, we obtain a Hamilton cycle if p is odd, and we obtain a cycle omitting only two vertices $12l^p$ and $21l^p$ if p is even.

We obtain case 2.2 in Lemma 2 if the above construction is repeated for paths d'_i ($0 \leqq i \leqq p$) defined as

$$d'_i = [l^i2l^{p-i}1, l^i2l^{p-i-1}1l, \cdots, l^i21l^{p-i}, l^i12l^{p-i}, \cdots, l^i1l^{p-i}2].$$

It remains only to exhibit i-edges in the constructed cycles. Each cycle contains edges $k_i = [l^i12l^{p-i}, l^i21l^{p-i}]$ for $1 \leqq i < p$. As a full set of i-edges, we take $e_1 = k_{p-1}$ and $e_i = k_{i-1}$ for $1 < i$. □

The graph $GE(Q|P)$ for two arbitrary posets P and Q consists of the subgraphs $GE(Q|L)$ ($L \in \mathscr{L}(P)$) connected to each other. All these subgraphs are isomorphic to $GE(Q|l^p)$, where $p = |P|$. We can therefore consider i-edges in $GE(Q|L)$ as those

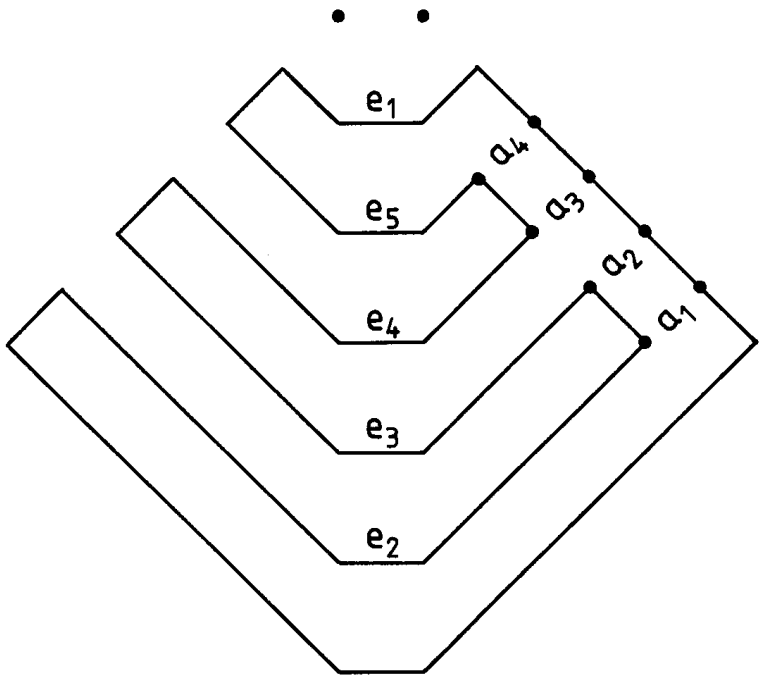

FIG. 1. *A long cycle defined in Lemma* 2, *case* 2.1, *for* $p = 6$.

having ith and $(i + 1)$th places of L adjacent. We can define analogously a full set of i-edges in $GE(Q|L)$.

We now prove a lemma similar to the result of Batagelj and Pisanski [1].

LEMMA 3. *Let a graph* $G = GE(Q|P)$ *contain a family* $\mathscr{C}$ *of cycles such that each* $GE(Q|L)$ *as a subgraph of* G *contains exactly one cycle of* $\mathscr{C}$. *If each of these cycles has a full set of* i*-edges corresponding to a fixed full set in* $GE(Q|l^p)$, *cycles in* $\mathscr{C}$ *can be combined into one cycle containing all their vertices.*

Proof. The graph $GE(P)$ is connected, so it contains a spanning tree, say T. To each edge $[L_1, L_2]$ of T we assign a number i $(1 \leqq i < p)$ such that L_1 can be obtained from L_2 by an elementary transformation on the ith and the $(i + 1)$th elements. Thus, the i-edges in the graphs $GE(Q|L_1)$ and $GE(Q|L_2)$ for a fixed i are parallel. We glue all such edges to obtain one cycle from the cycles of $\mathscr{C}$. These operations of gluing are possible because there is at most one edge labeled by i $(1 \leqq i < p)$ adjacent to a given vertex of $GE(P)$. □

Now we are ready to formulate the first step in the proof of the theorem.

LEMMA 4. *The graph* $G = GE(1|2|P)$ *contains a Hamilton cycle when* $|P|$ *is odd.*

Proof. The graph G consists of subgraphs $GE(1|2|L)$ $(L \in \mathscr{L}(P))$ isomorphic to $GE(1|2|l^p)$. Lemma 2, case 1 gives us Hamilton cycles in $GE(1|2|L)$ with full sets of i-edges. Using Lemma 3, we can connect all these cycles obtaining Hamilton cycle in $GE(1|2|P)$. □

Rewriting the same proof for p even is complicated by the lack of a Hamilton cycle (Lemma 12) in $GE(1|2|l^p)$. We eliminate this difficulty by first using Lemma 3 to connect cycles from Lemma 2 and next applying Lemma 5 to attach the remaining vertices to the obtained cycle.

LEMMA 5. *If* P *is not a chain, then* $GE(P)$ *can be split by removing some edges into paths of number of vertices greater than one.*

Proof. We proceed by induction on $p = |P|$. If $p \leqq 2$, then Lemma 5 is obviously true. Let us assume that it holds for all posets on $(p - 1)$-elements, where $p \geqq 3$ and let P be a p-element poset. There are two possible cases: P either contains or does not contain

a $(p-1)$-element chain. In the former case, $GE(P)$ is a path of number of vertices at least two. In the latter case, $GE(P)$ does not have such a path. Let $x \in P$ be a minimal element in P and consider the poset $R = P - x$. By the inductive assumption, R can be split into paths of number of vertices at least two. Hence the graph $GE(P)$, which consists of graphs $GE(P-x)$ for all minimal $x \in P$, can be split into such paths. □

We can now prove the following lemma.

LEMMA 6. *Let $G = GE(1|2|P)$ where $p = |P|$ is even. If P is a chain, then G contains a Hamilton path, and if P is not a chain, then G contains a Hamilton cycle.*

Proof. If $P = l^p$, then by case 2.1 of Lemma 2 we have in $GE(1|2|l^p)$ the edge $[12l^p, 21l^p]$ and a cycle containing the remaining vertices. The vertex $21l^p$ from the edge is a neighbor of the vertex $2l1l^{p-1}$ belonging to the cycle. So if we have given only this edge and cycle, then by adding the edge $[21l^p, 2l1l^{p-1}]$ and removing one of the edges adjacent to $2l1l^{p-1}$ from the cycle, we obtain a Hamilton path in G.

If P is not a chain, then we consider two cases: $p = 2$ and $p > 2$. For $p = 2$, the only poset to consider is $1|2|3|4$. In this case, a Hamilton cycle can be easily obtained (by the Steinhaus–Johnson–Trotter algorithm [5], [9], [10] or by applying Lemma 10 for $Q = 1|2|3$ and $P = \{4\}$). If $p > 2$, then by Lemma 5, $GE(P)$ can be split into a family of paths $\mathscr{D}$ of number of vertices at least two. Let $D = [L_{D(1)}, L_{D(2)}, \cdots, L_{D(s(D))}] \in \mathscr{D}$. Let $G_{D(i)}$ denote the graph $GE(1|2|L_{D(i)})$. In $G_{D(1)}$, considered as a subgraph of $GE(1|2|P)$, we take a cycle defined in the proof of Lemma 2 (case 2.2). In all other subgraphs $G_{D(i)}$, we take the cycles defined in the proof of case 2.1 of Lemma 2. Note that the cycle from case 2.1 must contain the edge $[l^p12, l^p21]$ and the cycle from case 2.2 must contain the edge $[12l^p, 21l^p]$ because of Lemma 1. We combine these cycles into one cycle applying Lemma 3, and then we replace the edges $[12L_{D(1)}, 21L_{D(1)}]$ by paths $[12L_{D(1)}, 12L_{D(2)}, \cdots, 12L_{D(s(D))}, 21L_{D(s(D))}, \cdots, 21L_{D(1)}]$ and the edges $[L_{D(2)}12, L_{D(2)}21]$ by paths $[L_{D(2)}12, L_{D(1)}12, L_{D(1)}21, L_{D(2)}21]$ (see Fig. 2). This results in a Hamilton cycle in G for $p \neq 2$. □

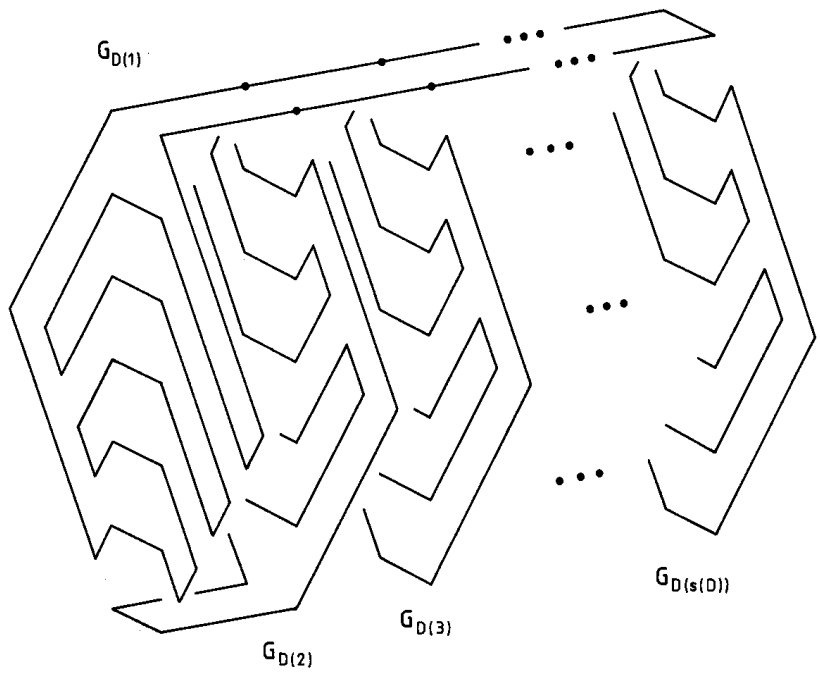

FIG. 2. *The method of connecting to long cycles in $G_{D(i)}$ vertices, which are absent in these cycles, to obtain a Hamilton cycle, as in Lemma 6.*

Lemmas 4 and 6 constitute a proof of the theorem for Q consisting of two isolated elements. This result was proved for not necessarily adjacent transpositions by Göbel [4].

3. The theorem for an arbitrary Q. We first provide another proof of a useful lemma from [2].

LEMMA 7. *The graph* $G = GE((1|2)(k^q|l^p))$ *contains a Hamilton cycle for every* $p, q > 0$.

Proof. We proceed by induction on p. If $p = 1$, then the cycle is of the form $[12k^ql, 12k^{q-1}lk, \cdots, 12lk^q, 21lk^q, \cdots, 21k^ql]$. Let us assume that Lemma 7 holds for $p-1$ and split G into graphs $G_i = GE((1|2)k^{q-i}l(k^i|l^{p-1}))$ $(0 \leqq i \leqq q)$. By the inductive hypothesis, each G_i for $i > 0$ has a Hamilton cycle that contains the following edges:

$$a_i = [21k^{q-i}lk^il^{p-1}, 12k^{q-i}lk^il^{p-1}]$$

and

$$b_i = [12k^{q-i}lk^il^{p-1}, 12k^{q-i}lk^{i-1}lkl^{p-2}],$$

because vertex $12k^{q-i}lk^il^{p-1}$ is of degree 2 in G_i. Then, replacing the edge a_1 in G_1 by the path $[21k^{q-1}lkl^{p-1}, 21k^ql^p, 12k^ql^p, 12k^{q-1}lkl^{p-1}]$ we join G_0 to G_1. Finally, we glue the edges b_1 and b_2, b_3 and b_4, $\cdots$ and a_2 and a_3, a_4 and a_5, $\cdots$ to obtain a Hamilton cycle in G. □

Now we use Lemma 7 to prove the following lemma.

LEMMA 8. *The graph* $G = GE(((1|2)k^q)|l^p)$ *contains a Hamilton cycle for every* $p, q > 0$.

Proof. We split G into subgraphs

$$G_i = GE((1|l^{p-i})2(k^q|l^i),(2|l^{p-i})1(k^q|l^i)) \qquad (0 \leqq i \leqq p)$$

each of which will be shown to contain a Hamilton path with end vertices $x_i = 1l^{p-i}2l^ik^q$ and $y_i = 2l^{p-i}1l^ik^q$. In particular, G_0 contains the path

$$[1l^p2k^q, l1l^{p-1}2k^q, \cdots, l^p12k^q, l^p21k^q, \cdots, 2l^p1k^q].$$

For $i > 0$ we again split G_i into subgraphs G_{ij} $(0 \leqq j \leqq p-i)$ of the form

$$G_{ij} = GE(l^{2j}(1|l)l^{p-i-2j-1}2(l^i|k^q)) \quad \text{for } 0 \leqq j < (p-i)/2,$$

$$G_{ij} = GE(l^{p-i}(1|2)(l^i|k^q)) \quad \text{for } j = (p-i)/2,$$

$$G_{ij} = GE(l^{2(p-i-j)+1}(l|2)l^{i+2j-p-1}1(l^i|k^q)) \quad \text{for } (p-i)/2 < j \leqq p-i.$$

By Lemma 7, each G_{ij} (as isomorphic to $GE((1|2)(l^i|k^q))$) has a Hamilton cycle that contains the edge $[x_{ij}, y_{ij}]$, where

$$x_{ij} = l^{2j}1l^{p-i-2j}2l^ik^q \quad \text{for } 0 \leqq j < (p-i)/2,$$

$$x_{ij} = l^{p-i}12l^ik^q \quad \text{for } j = (p-i)/2,$$

$$x_{ij} = l^{2(p-i-j)}2l^{i+2j-p}1l^ik^q \quad \text{for } (p-i)/2 < j \leqq p-i,$$

and

$$y_{ij} = l^{2j+1}1l^{p-i-2j-1}2l^ik^q \quad \text{for } 0 \leqq j < (p-i)/2,$$

$$y_{ij} = l^{p-i}21l^ik^q \quad \text{for } j = (p-i)/2,$$

$$y_{ij} = l^{2(p-i-j)-1}2l^{i+2j-p+1}1l^ik^q \quad \text{for } (p-i)/2 < j \leqq p-i.$$

By the same argument, G_{i0} contains the edge $a_i = [1l^{p-i}2l^ik^q, 1l^{p-i}2l^{i-1}klk^{q-1}]$. We remove edges $[x_{ij}, y_{ij}]$ from cycles in G_{ij} and then we add edges $[y_{ij}, x_{i(j+1)}]$ to obtain the required Hamilton path in G_i. Finally, we connect x_0 and x_1, x_2 and x_3, $\cdots$ y_0 and y_1, y_2 and y_3, $\cdots$ to obtain $\lfloor (p+1)/2 \rfloor$ cycles in G. If p is even, then there remains a path in G_p and we connect its end vertices to obtain the last cycle. By gluing a_1 and a_2, a_3 and a_4, $\cdots$, we obtain a Hamilton cycle in G. See Fig. 3. □

LEMMA 9. *The graph* $G = GE((k^r(1|2)k^s)|l^p)$ *contains a Hamilton cycle when* $p, r+s>0$.

Proof. If $s = 0$, then the graph G is isomorphic to $G = GE(((1|2)k^r)|l^p)$ and has a Hamilton cycle by Lemma 8. For $s > 0$ we proceed by induction on r. If $r = 0$, then our task can be reduced to Lemma 8. We divide G into subgraphs $G_i = GE(l^{p-i}k((k^{r-1}(1|2)k^s)|l^i))$ for $0 \leqq i \leqq p$. By the inductive hypothesis, there exists a Hamilton cycle C_i in every G_i for $i > 0$. Each cycle C_i contains the edges

$$a_i = [l^{p-i}kl^ik^{r-1}12k^s, l^{p-i}kl^ik^{r-1}21k^s],$$

and, for $r = 1$,

$$b_i = [l^{p-i}kl^i12k^s, l^{p-i}kl^{i-1}1l2k^s],$$

for $r > 1$,

$$b_i = [l^{p-i}kl^ik^{r-1}12k^s, l^{p-i}kl^{i-1}klk^{r-2}12k^s]$$

because vertex $l^{p-i}kl^ik^{r-1}12k^s$ is of degree 2.

In the cycle C_1 we exchange the edge a_1 for the path $[l^{p-1}klk^{r-1}12k^s, l^pk^r12k^s, l^pk^r21k^s, l^{p-1}klk^{r-1}21k^s]$ connecting G_0 to the cycle in G_1. Then we glue b_1 with b_2, b_3 with b_4, and so on, a_2 with a_3, a_4 with a_5, and so on, and we obtain a Hamilton cycle in G. □

LEMMA 10. *If* $q = |Q| > 2$, $|\mathscr{L}(Q)|$ *is even and* $GE(Q)$ *contains a Hamilton path and* $p > 0$ *then* $GE(Q|l^p)$ *has a Hamilton cycle.*

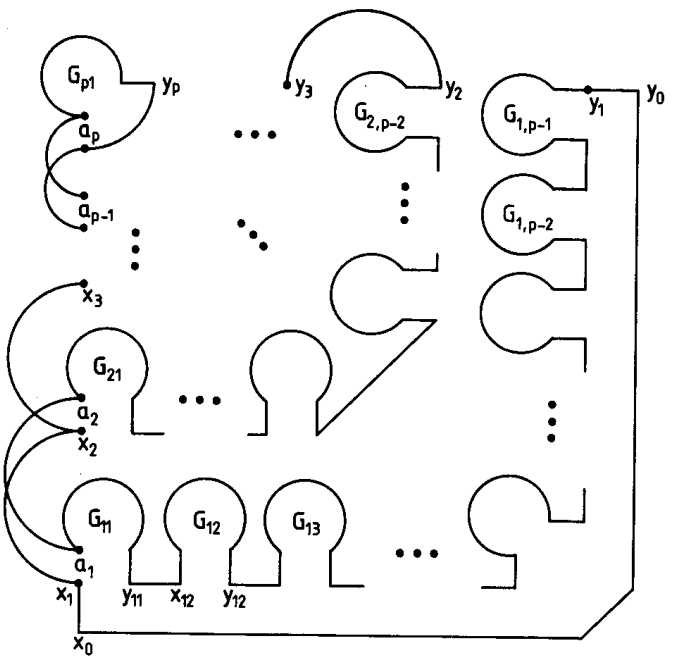

FIG. 3. *A Hamilton cycle defined in Lemma 8 for even p.*

Proof. Let $[K_1, K_2, \cdots, K_{2n}]$ be a Hamilton path in $GE(Q)$. We split $GE(Q|l^p)$ into n subgraphs $G_i = GE(K_{2i-1}|l^p, K_{2i}|l^p)$; $1 \leqq i \leqq n$. G_i is isomorphic to $GE((k^r(1|2)k^s)|l^p)$ for some r and s. Thus G_i has a Hamilton cycle C_i by Lemma 9. Let $K_j = k_{j1}k_{j2}\cdots k_{jq}$, and denote the edges

$$a_j = [l^p k_{j1}\cdots k_{jq}, l^{p-1}k_{j1}lk_{j2}\cdots k_{jq}]$$

and

$$b_j = [k_{j1}\cdots k_{jq}l^p, k_{j1}\cdots k_{j,q-1}lk_{jq}l^{p-1}].$$

Since $\deg(l^p k_1 \cdots k_q) = \deg(k_1 \cdots k_q l^p) = 2$, by Lemma 1 the edges a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i} belong to C_i for $1 \leqq i \leqq n$. If K_j and K_{j+1} do not differ in the first pair of elements, then b_j and b_{j+1} are parallel. In the other case, a_j and a_{j+1} are parallel. We then glue parallel edges a_j and a_{j+1} or b_j and b_{j+1} of G_j and G_{j+1} for $j = 2i$ $(1 \leqq i < n)$ obtaining a Hamilton cycle in $GE(Q|l^p)$. See Fig. 4. □

LEMMA 11. *If $q = |Q| > 2$, $p = |P| > 0$ and $GE(Q)$ has an even number of vertices and contains a Hamiton path then $GE(Q|P)$ has a Hamilton cycle.*

Proof. We use Lemma 3 to build a Hamilton cycle in $GE(Q|P)$ from Hamilton cycles in $GE(Q|L)$ $(L \in \mathscr{L}(P))$. These cycles exist by Lemma 10, because $GE(Q|L)$ are isomorphic to $GE(Q|l^p)$. There remains to find a full set of i-edges in $GE(Q|l^p)$. Let $K = k_1k_2\cdots k_q$ be a linear extension of Q. The vertex $x_i = l^i k_1\cdots k_q l^{p-i}(2 \leqq i \leqq p)$ is a linear extension of Q and has two neighbors in a Hamilton cycle in $GE(Q|l^p)$. At most one of these neighbors can be obtained from x_i by transposition of the ith and $(i+1)$th elements. Moreover, there exists an edge incident to x_i in $GE(Q|l^p)$ not equivalent to this transposition, and we take it as the $(i-1)$-edge e_{i-1}. This way, we obtain i-edges for $1 \leqq i \leqq p-1$. □

Proof of the theorem. For $P \neq \varnothing$ the theorem is obvious. For $|Q| = 2$, it follows from Lemmas 4 and 6, and for $|Q| > 2$, it follows from Lemma 11. □

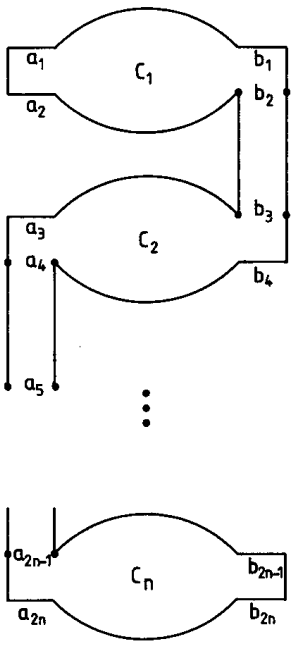

FIG. 4. *A Hamilton cycle constructed in proof of Lemma* 10.

Note that in some cases the lemmas give us more information than the theorem requires. Namely, a graph of type $GE(Q|P)$, where $GE(Q)$ has a Hamilton path and an even number of vertices, contains in many cases not only a Hamilton path, but also a Hamilton cycle. The only exceptions are graphs $GE(Q)$ and $GE((1|2)|l^p)$ where p is even. For the latter case, we prove the following result, which was originally proved by Ruskey [8].

LEMMA 12. *The graph* $G = GE((1|2)|l^p)$, *where* p *is even, has no Hamilton cycle.*

Proof. We can decompose G into two subgraphs $G_1 = GE(12|l^p)$ and $G_2 = GE(21|l^p)$.

These parts are isomorphic to each other. The graph G_i $(i = 1, 2)$ is bipartite, and we can show that $d(G_i)$ is equal to $p/2 + 1$. We proceed by induction on p. If p is equal to 0, then G_i has only one linear extension and $d(G_i) = 1$. If for $p - 2$, $d(G_i) = p/2$, then we can split G_1 for p into three parts (analogous consideration can be carried out for G_2): $H = GE((12|l^{p-2})l^2)$, which is isomorphic to G_1 for $p - 2$; $I = GE((1|l^{p-1})(2|l))$ such that $d(I) = 0$; $J = \mathrm{GE}(l^p12)$, which adds 1 to the result for $p - 2$ finishing our inductive consideration.

As we know, $d(G) = 0$. It follows from that fact that each bipartition of G consists of the greater set in the bipartition of one G_i and the smaller one of the second G_i. There are $p + 1$ edges $a_i = [l^i12l^{p-i}, l^i21l^{p-i}]$ between G_1 and G_2 in G. If C is a Hamilton cycle in G, then C can be split into at most p maximal paths such that each of them is contained in one G_i because every two successive paths must be joined by an edge a_j for some $0 \leqq j \leqq p$ and there are $p + 1$ such edges. Moreover, only p of these edges can be used by C since a cycle must contain an even number of a_i's.

Thus, there are at most $p/2$ paths in each G_i that cover all their vertices. This is, however, impossible because $d(G_i) = p/2 + 1$ $(i = 1, 2)$ and at least this number of paths is necessary to cover G_i for $i = 1, 2$. □

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REFERENCES

[1] V. BATAGELJ AND T. PISANSKI, *Hamilton cycles in the cartesian product of a tree and a cycle*, Discrete Math., 38 (1982), pp. 311–312.

[2] M. BUCK AND D. WIEDEMAN, *Gray codes with restricted density*, Discrete Math., 48 (1984), pp. 163–171.

[3] P. EADES, M. HICKEY, AND R. READ, *Some Hamilton paths and a minimal change algorithms*, J. Assoc. Comput. Mach., 31 (1984), pp. 19–29.

[4] F. GÖBEL, *On a problem of F. Ruskey*, Memorandum No. 832, Twente University of Technology, Faculty of Applied Mathematics, Enschede, the Netherlands, 1989.

[5] S. M. JOHNSON, *Generation of permutations by adjacent transpositions*, Math. Comp., 17 (1963), pp. 282–285.

[6] C. W. KO AND F. RUSKEY, *Solution of some multidimensional lattice path parity difference recurrence relations*, Discrete Math., 71 (1988), pp. 47–56.

[7] F. RUSKEY, *Adjacent interchange generation of combinations*, J. Algorithms, 9 (1988), pp. 162–180.

[8] ———, *Generating the linear extensions of posets by transpositions*, J. Combin. Theory Ser. B, to appear.

[9] H. STEINHAUS, *One Hundred Problems in Elementary Mathematics*, Dover, New York, 1979.

[10] H. F. TROTTER, *Algorithm* 115: PERM, Comm. ACM, 5 (1962), pp. 434–435.