Algebraic theories and monads

Homework

# Equational theories

Let $\mathscr{X} = \{x, y, z, \ldots\}$ be a countably infinite set. A *signature* $\Sigma$ is a non-empty finite set of function symbols with arities (which we write in superscript). A set of all $\Sigma$-terms with variables from $\mathscr{X}$ is denoted $\mathcal{T}_\Sigma \mathscr{X}$.

An *equational theory* $T$ is a pair $\langle \Sigma, E \rangle$, where $\Sigma$ is a signature, and $E \subseteq \mathcal{T}_\Sigma \mathscr{X} \times \mathcal{T}_\Sigma \mathscr{X}$ is a finite set of pairs of $\Sigma$-terms with variables from $\mathscr{X}$, which we intuitively interpret as equations, and denote as $t_0 = t_1$. For example, the theory of monoids $\mathbf{Mon} = \langle \Sigma^{\mathbf{Mon}}, E^{\mathbf{Mon}} \rangle$ can be given as follows:

$$\begin{aligned}
\Sigma^{\mathbf{Mon}} &= \{\cdot^{(2)}, \varepsilon^{(0)}\} \\
E^{\mathbf{Mon}} &= \{(x \cdot y) \cdot z = x \cdot (y \cdot z), \quad \varepsilon \cdot x = x, \quad x \cdot \varepsilon = x\}
\end{aligned}$$

This reads that **Mon** consists of a binary symbol $\cdot$ and a nullary symbol $\varepsilon$, such that $\cdot$ is associative, and $\varepsilon$ is both a left and a right unit of $\cdot$.

For an equational theory $T = \langle \Sigma, E \rangle$, we define a relation $\approx_T \subseteq \mathcal{T}_\Sigma \mathscr{X} \times \mathcal{T}_\Sigma \mathscr{X}$ ('equality in one step') as the smallest relation generated by the following rules:

$$\frac{}{t_0\theta \approx_T t_1\theta} \quad \text{for any substitution } \theta \text{ if } (t_0 = t_1) \in E \text{ or } (t_1 = t_0) \in E$$

$$\frac{t_k \approx_T t'_k}{f(t_1, \ldots, t_k, \ldots, t_n) \approx_T f(t_1, \ldots, t'_k, \ldots, t_n)} \quad \text{for } f^{(n)} \in \Sigma \text{ and } k \in \{1 \ldots n\}$$

By $\approx^*_T \subseteq \mathcal{T}_\Sigma \mathscr{X} \times \mathcal{T}_\Sigma \mathscr{X}$ we denote the reflexive and transitive closure of $\approx_T$.

# Problem 1

Consider the following theory **Cut**:

$$\begin{aligned}
\Sigma^{\mathbf{Cut}} &= \Sigma^{\mathbf{Mon}} \cup \{\text{-}^{*(1)}\} \\
E^{\mathbf{Cut}} &= E^{\mathbf{Mon}} \cup \{x^\bullet \cdot y = x^\bullet, \quad x \cdot y^\bullet = (x \cdot y)^\bullet\}
\end{aligned}$$

Show that $-^\bullet$ is idempotent, that is, for all $x \in \mathscr{X}$, it is the case that

$$(x^\bullet)^\bullet \approx^*_{\mathbf{Cut}} x^\bullet$$

## Rewriting

A term-rewriting system is similar to an equational theory, but we interpret the 'equations' as directed rewrite rules. In detail, a *term-rewriting system* is a pair $W = \langle \Sigma, R \rangle$, where $\Sigma$ is a signature, and $R$ is a finite set of pairs of $\Sigma$-terms, written as $t_0 \mapsto t_1$. For example:

$$\begin{aligned} \Sigma^{\mathbf{RMon}} &= \{\cdot^{(2)}, \varepsilon^{(0)}\} \\ R^{\mathbf{RMon}} &= \{(x \cdot y) \cdot z \mapsto x \cdot (y \cdot z), \quad \varepsilon \cdot x \mapsto x, \quad x \cdot \varepsilon \mapsto x\} \end{aligned}$$

We define a relation $\rightsquigarrow_W \subseteq \mathcal{T}_\Sigma \mathscr{X} \times \mathcal{T}_\Sigma \mathscr{X}$ ('one step of rewriting') as the smallest relation generated by the following rules:

$$\frac{}{t_0\theta \rightsquigarrow_W t_1\theta} \quad \text{for any substitution } \theta \text{ if } (t_0 \mapsto t_1) \in R$$

$$\frac{t_k \rightsquigarrow_W t'_k}{f(t_1, \ldots, t_k, \ldots, t_n) \rightsquigarrow_W f(t_1, \ldots, t'_k, \ldots, t_n)} \quad \text{for } f^{(n)} \in \Sigma \text{ and } k \in \{1 \ldots n\}$$

By $\rightsquigarrow^*_W \subseteq \mathcal{T}_\Sigma \mathscr{X} \times \mathcal{T}_\Sigma \mathscr{X}$ we denote the reflexive and transitive closure of $\rightsquigarrow_W$.

A term $t$ is a *normal form* if there exists no term $t'$ such that $t \rightsquigarrow_W t'$.

A term-rewriting system is *confluent* if for all terms $t_0$, $t_1$, and $t_2$ such that $t_0 \rightsquigarrow_W t_1$ and $t_0 \rightsquigarrow_W t_2$, there exists a term $t_3$ such that $t_1 \rightsquigarrow^*_W t_3$ and $t_2 \rightsquigarrow^*_W t_3$.

We say that a system is *normalising* if it is confluent and every term has a normal form. Note that in a normalising system, every term $t$ has exactly one corresponding normal form, which we denote $[t]$.

Given an equational theory $T = \langle \Sigma^T, E^T \rangle$ and a term-rewriting system $W = \langle \Sigma^W, R^W \rangle$, we say that $W$ is a *directionalisation* of $T$ if $\Sigma^W = \Sigma^T$ and $(t_0 = t_1) \in E^T$ if and only if either $(t_0 \mapsto t_1) \in R^W$ or $(t_1 \mapsto t_0) \in R^W$ (but not both).

## Problem 2

Let $W = \langle \Sigma, R \rangle$ be a normalising directionalisation of an equational theory $T = \langle \Sigma, E \rangle$. Show that $t_0 \approx^*_W t_1$ if and only if $[t_0] = [t_1]$. (The latter is the usual equality on terms. Intuitively: to check that two terms are equal in a theory, it is enough to normalise them and syntactically compare the normal forms.)

## Models and free models

Let $T = \langle \Sigma, E \rangle$ be an equational theory. A $\Sigma$-structure $\mathfrak{A}$ consists of the following elements:

- a set $A$,
- for each $f^{(n)} \in \Sigma$, a function $[\![f]\!]^{\mathfrak{A}} : A^n \to A$.

We write $\mathfrak{A} = \langle A, [\![\text{-}]\!]^{\mathfrak{A}} \rangle$.

Given a $\Sigma$-structure $\mathfrak{A}$ as above and a function $\sigma : \mathscr{X} \to A$ ('valuation of variables'), we define the valuation of $\Sigma$-terms as follows:

- $[\![x]\!]^{\mathfrak{A}}_{\sigma} = \sigma(x)$,
- $[\![f(t_1, \ldots, t_n)]\!]^{\mathfrak{A}}_{\sigma} = [\![f]\!]^{\mathfrak{A}}([\![t_1]\!]^{\mathfrak{A}}_{\sigma}, \ldots, [\![t_n]\!]^{\mathfrak{A}}_{\sigma})$ for each $f^{(n)} \in \Sigma$.

A *model* of $T$ is a $\Sigma$-structure $\mathfrak{A}$ such that for all valuations $\sigma$ and $(t_0 = t_1) \in E$, it is the case that $[\![t_0]\!]^{\mathfrak{A}}_{\sigma} = [\![t_1]\!]^{\mathfrak{A}}_{\sigma}$.

A *homomorphism* between models $\mathfrak{A} = \langle A, [\![\text{-}]\!]^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle B, [\![\text{-}]\!]^{\mathfrak{B}} \rangle$ is a function $h : A \to B$ such that for all $f^{(n)} \in \Sigma$ and $a_1, \ldots, a_n \in A$, it is the case that:

$$h([\![f]\!]^{\mathfrak{A}}(a_1, \ldots, a_n)) = [\![f]\!]^{\mathfrak{B}}(h(a_1), \ldots, h(a_n))$$

A *free model* consists of the following elements:

- For each set $X$ (the set of 'generators'), a model of $T$, which we denote $\mathcal{F}X = \langle FX, [\![\text{-}]\!]^{\mathcal{F}X} \rangle$,
- A family of functions $\eta_X : X \to \mathcal{F}X$,

such that for any model $\mathfrak{A} = \langle A, [\![\text{-}]\!]^{\mathfrak{A}} \rangle$ and a function $g : X \to A$, there exists a unique homomorphism $\widehat{g} : FX \to A$ between $\mathcal{F}X$ and $\mathfrak{A}$ such that

$$g = \widehat{g} \circ \eta_X$$

Note that a free model gives us a monad. In detail:

- The assignment $X \mapsto FX$ is functorial. The action on morphisms is given as $(f : X \to Y) \mapsto \widehat{\eta_Y \cdot f} : FX \to FY$.
- The unit of the monad ('return' in Haskell) is given by $\eta$.
- The multiplication ('join' in Haskell) is given by $\widehat{\mathsf{id}_{FX}} : FFX \to FX$.

## Problem 3

For $n \in \mathbb{N}$, we define $\underline{n} = \{1, \ldots, n\}$. We define the following equational theory, which models mutable memory cell with $n$ possible values:

$$
\begin{aligned}
\Sigma^{\mathbf{St}} &= \{\mathsf{get}^{(n)}, \mathsf{put}_1^{(1)}, \ldots, \mathsf{put}_n^{(1)}\} \\
E^{\mathbf{St}} &= \{\mathsf{get}(x_1, \ldots, x_{k-1}, \mathsf{get}(y_1, \ldots, y_k, \ldots, y_n), x_{k+1}, \ldots, x_n) \\
&\qquad\quad = \mathsf{get}(x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_n), \\
&\qquad \mathsf{put}_k(\mathsf{put}_j(x)) = \mathsf{put}_j(x), \\
&\qquad \mathsf{put}_k(\mathsf{get}(x_1, \ldots, x_k, \ldots x_n)) = \mathsf{put}_k(x_k), \\
&\qquad \mathsf{get}(x_1, \ldots, x_{k-1}, \mathsf{put}_k(x_k), x_{k+1}, \ldots, x_n) \\
&\qquad\quad = \mathsf{get}(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n)\}
\end{aligned}
$$

(Note that the above definition of $E^{\mathbf{St}}$ contains schemes of equations. Each entry should be interpreted as a set of equations for $j, k \in \underline{n}$.)

Prove that the following is the free model of **St**:

- $A \mapsto (A \times \underline{n})^{\underline{n}}$
- $[\![\mathsf{get}]\!](f_1, \ldots, f_n) = \lambda(s \in \underline{n}).f_s(s)$
- $[\![\mathsf{put}_k]\!](f) = \lambda(s \in \underline{n}).f(k)$
- $\eta_A(a) = \lambda(s \in \underline{n}).\langle a, s\rangle$

Show that the monad given by the free model of **St** is the state monad as known from Haskell.