Compressed Membership for NFA (DFA) with Compressed Labels is in NP (P)

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Straight Line Programms SLPs

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Context free grammar defining a single word. (Chomsky normal form).

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$$A_1 \to a \quad A_2 \to A_1 A_1 \; \dots \; A_{\ell+1} \to A_\ell A_\ell \qquad A \to A_{\ell_0} A_{\ell_1} \dots$$

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SLPs as a compression model

- application (LZ, logarithmic transformation)
- theory (formal languages)
- up to exponential compression
- preserves/captures word properties

Usage and work on SLP

Theory

- word equations (Plandowski: satisfiability in PSPACE)

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- equality
- pattern matching

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LZW/LZ dealing algorithms

- $\mathcal{O}(n \log n)$ pattern matching for LZ compressed text
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Independent interest

- indexing structure for SLP

Compressed membership

- SLPs are used
- membership problem
- develop tools/gain understanding

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Compressed membership [Plandowski & Rytter; *Jewels are forever* 1999]

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Known results

RE, CFG, Conjunctive grammars . . .

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Open questions

- Posted in *Jewels are forever*
- some solved
- Compressed membership for NFA

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Input: SLP, NFA N
Output: Yes/No

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- SLP for w
- NFA N, compressed transitions

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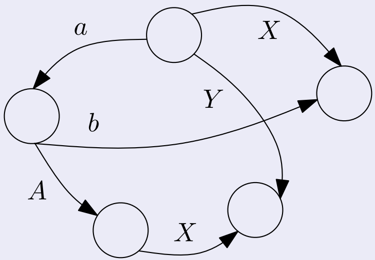

Compressed membership for NFA: complexity

Complexity

- NP-hardness (subsum), already for
  - acyclic NFA
  - unary alphabet
- in PSPACE: enough to store positions inside decompressed words

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Conjecture

In NP.

Partial results

- Plandowski & Rytter (unary in NP)
- Lohrey & Mathissen (highly periodic in NP, highly aperiodic in P)

New results

Theorem

Fully compressed membership for NFA is in NP.

Theorem

Fully compressed membership for DFA is in P.

New results

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Theorem

Fully compressed membership for DFA is in P.

New technique

New, interesting technique.

Convention

Convention

- SLPs given as a single grammar
- $X_i \to X_j X_k$, implies $i > j$, $i > k$
- input word: X_n
- word(X_i)

Idea: Recompression

Difficulty: the words are long. Shorten them.

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Deeper understanding

New production: $d \to ab$. Building new SLP (recompression).
SLP problems: hard, as SLP are different.
Building canonical SLP for the instance.

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What to do with a^n?

$a \quad a \quad c \quad a \quad a \quad a$

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What to do with a^n? Replace each non-extendible a^n by a single symbol.

$a_2 \quad c \quad a_3$

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What to do with a^n? Replace each non-extendible a^n by a single symbol.

$a_2 \quad c \quad a_3$

Problems

- easy for text, what about grammar?
- what to do with the NFA?

Local decompression

Re-compression

- decompressed text: easy; size: large,
- compressed text: hard; size: small.

Local decompression

Re-compression

- decompressed text: easy; size: large,
- compressed text: hard; size: small.

Local decompression

Decompress locally the SLP:

$$X \to uYvZ$$

- u, v: blocks of letters, linear size
- Y, Z: nonterminals
- recompression inside u, v

Outline

Outline of the algorithm

while $|\text{word}(X_n) > n|$ **do**
  $L_\Sigma \leftarrow$ list of letters, $L_P \leftarrow$ list of pairs
  for $ab \in L_P$ **do**
    compress pair ab, modify N accordingly
  for $a \in L_\Sigma$ **do**
    compress a non-extendible appearances, modify N accordingly

Decompress the word and solve the problem naively.

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Decompress the word and solve the problem naively.

Theorem

There are $\mathcal{O}(\log \text{word}(X_n))$ *iterations.*

Proof.

Consider two consecutive letters ab. One of them is compressed. So word shortens by a constant factor. □

Formalisation

New symbols: letters.

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Grammar invariants

- the set of nonterminals is $\{X_n, \ldots, X_1\}$ (the input ones)
- the productions are of the form

$$X_i \to uX_jvX_k \quad \text{or} \quad X_i \to uX_jv \quad \text{or} \quad X_i \to u,$$

- for $X_i \to uX_jvX_k$, the input had a production $X_i \to X_jX_k$

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In this way the grammar does not blow up (the skeleton is the same).

NFA invariants

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NFA invariants

- transitions of N:

  letter transitions labelled by a single letter

  nonterminal transition labelled by nonterminal
- nonterminal transition have counterparts in the input NFA
  - the same nonterminal
  - corresponding start and end
  - the same multiplicity

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- nonterminal transition have counterparts in the input NFA
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The nonterminal part is 'the same' (of size n).

What is hard, what is easy

What is hard to compress, what easy?

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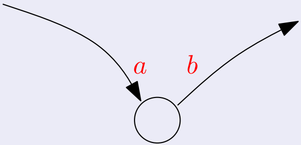
What is hard to compress, what easy?

Hard

- a letter a is outer, if it is the first or last letter of some word(X_i)
- a pair ab is crossing if
  - $X_i \to uaX_jvX_k$, where word$(X_j) = b \dots$
  - ab spreads over transitions
    at least one nonterminal



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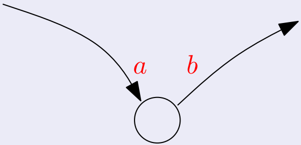

Easy

- a letter a is inner otherwise
- a pair ab is non-crossing otherwise

A little detailed outline

Detailed outline

while $|\text{word}(X_n) > n|$ **do**
  while possible **do**
    for a: inner letter **do**
      compress appearances of a
    for non-crossing pair ab in $\text{word}(X_n)$ **do**
      compress ab

A little detailed outline

Detailed outline

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  while possible **do**
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    for non-crossing pair ab in $\text{word}(X_n)$ **do**
      compress ab
  $L \leftarrow$ list of all outer letters
  for $a \in L$ **do**
    compress appearances of a
    for each $a_k b$ in $\text{word}(X_n)$ **do**
      compress $a_k b$

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while $|\text{word}(X_n) > n|$ **do**
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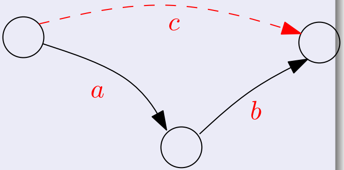

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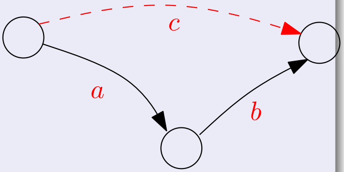

Lemma

It works.

Proof.

The pair is non-crossing: it always appears inside production. □

Inner letter compression

Appearance compression for and inner letter a

compute the lengths $\ell_1, \ldots, \ell_k$ of a's non-extendible appearance
for each a^{ℓ_m} **do**
  for each production $X_i \to uX_jvX_k$ **do**
    replace non-extendible a^{ℓ_m} in in u, v by a_{ℓ_m}

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  for states p, q in N **do**
    guess, whether $\delta_N(p, a^{\ell_m}, q)$
    if guess is positive **then**
      verify the guess
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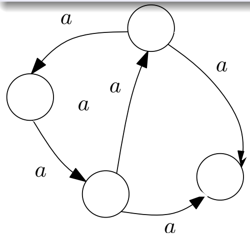

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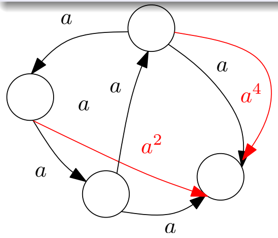

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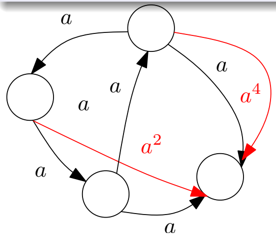

Lemma

It works.

- how many lengths: inner letter
- verification: Plandowski & Rytter result

Crossing pairs and outer letters

Lemma

$\mathcal{O}(n)$ outer letters and $\mathcal{O}(n^2)$ crossing pairs appearing in $\text{word}(X_n)$

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Proof.

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$\mathcal{O}(n)$ outer letters and $\mathcal{O}(n^2)$ crossing pairs appearing in $\text{word}(X_n)$

Proof.

- charge outer letter to the non-terminal
- pairs in $\text{word}(X_n)$:
  - appears is some explicit word $uX_j vX_k$, total size $\mathcal{O}(n^2)$.
  - spreads over $u\,\text{word}(X_j)$,
    charged to a production $X_i \to uX_j vX_k$, total size $\mathcal{O}(n)$. □

Convert hard to easy

Convert outer letters to inner, crossing pairs to non-crossing (Sequentially).

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Convert outer letters to inner, crossing pairs to non-crossing (Sequentially).

fix an outer letter a
convert it to inner
compress a appearances
make each pair of the form $a_\ell b$ non-crossing
compress each such pair

Turning a into an inner letter

- Cut a-prefix and a-suffix from each nonterminal.
- Represent word(X_i) as $a^{\ell_i} w a^{r_i}$, turn it into w.
  - $X_i \to uX_j vX_k$
  - $\text{word}(X_i) = u\,\text{word}(X_j)\,v\,\text{word}(X_k)$

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Changing an outer letter a to an inner one

for $i = 1 \,..\, n$ **do**
  let $X_i \to u X_j v X_k$
  replace $X_i \to u a^{\ell_j} X'_j a^{r_j} v a^{\ell_k} X'_k a^{r_k}$
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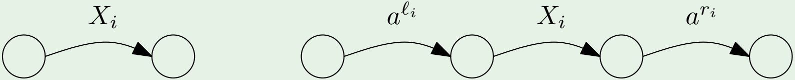
Lemma

The algorithm makes a an inner letter.

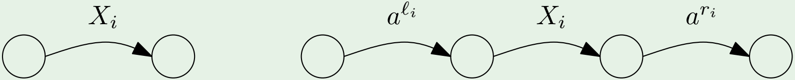
Adjusting the NFA

Adjust the transitions $\delta(p, X_i, q)$



Adjusting the NFA

Adjust the transitions $\delta(p, X_i, q)$



Lemma

This adjusts the NFA properly.

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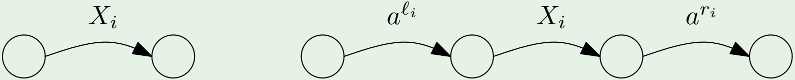

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a^ℓ

a^ℓ in NFA and grammar.

Adjusting the NFA

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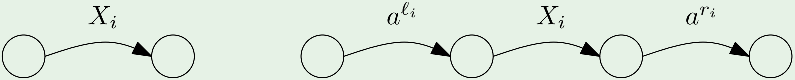

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This adjusts the NFA properly.

a^ℓ

a^ℓ in NFA and grammar.

Appearance compression

- guess and verify, whether $\delta_N(p, a^\ell, q)$
- by Plandowski & Rytter result (unary case in NP)

Crossing to non-crossing

Turn pairs $a_\ell b$ into non-crossing.

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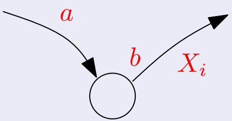
Turn pairs $a_\ell b$ into non-crossing.

- a_ℓ is inner
- 'pop' first letter of each non-terminal
- replace: $\text{word}(X_i) = bu \mapsto \text{word}(X_i) = u$



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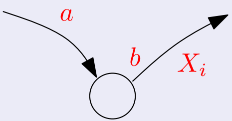

Lemma

After popping letters, no pair $a_\ell b$ is crossing.

Proof.

Easy, but goes into details. □

Sizes and running time

Running time

All algorithms run in time $npoly(n, |G|, |\Sigma|, |N|)$.

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Size of G

$$abbbcceaX_j addfeaaf X_k$$

In each iteration

Sizes and running time

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Size of G

$$abbbccea\color{red}{bha}\color{black}X_j\color{red}{ab}\color{black}addfeaaf\,\color{red}{cda}\color{black}X_k$$

In each iteration

- $\mathcal{O}(n)$ new letters

Sizes and running time

Running time

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Size of G

$$abbbcceabhaX_jabaddfeaafcdaX_k$$

In each iteration

- $\mathcal{O}(n)$ new letters
- shrinking by a constant factor

Sizes and running time

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All algorithms run in time $npoly(n, |G|, |\Sigma|, |N|)$.

Size of G

$$uv\,bhaX_j ab xyz\,cdaX_k$$

In each iteration

- $\mathcal{O}(n)$ new letters
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New letters ($|\Sigma|$)

- noncrossing pairs, inner letters appearance compression (shrinks $|G|$)
- outer letters and crossing pairs:
  there are $\mathcal{O}(n)$ outer letters and $\mathcal{O}(n^2)$ pairs in word(X_n)

Proof end

NFA size

- size $\mathcal{O}(|Q| \cdot |\Sigma| + n)$
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And this is it.

DFA

DFA vs NFA

Non-determinism: guessing and verifying transitions $\delta(p, a^\ell, q)$.

- for DFA: easy
- operations preserve determinism of DFA