Equations over sets of integers

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Language equations

$$\begin{cases} \varphi_1(X_1, \ldots, X_n) & = & \psi_1(X_1, \ldots, X_n) \\ & \vdots & \\ \varphi_m(X_1, \ldots, X_n) & = & \psi_m(X_1, \ldots, X_n) \end{cases}$$

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- φ_i: variables, constants, operations on languages.

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$L \subseteq \Omega^*$ is given by unique solution of a system with $\{\cup, \cap, \sim, \cdot\}$ and equations of the form $\varphi(X_1, \ldots, X_n) = \psi(X_1, \ldots, X_n)$

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Theorem (Kunc STACS 2005)

There exists a finite L such that the greatest solution of

$$LX = XL$$

for $X \subseteq \{a, b\}^*$ is co-recursively enumerable-hard.

Unary languages as sets of numbers

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$S \subseteq \mathbb{N}$ is given by unique solution of a system with $\{\cup, +\}$ or $\{\cap, +\}$ and equations of the form $\varphi(X_1, \ldots, X_n) = \psi(X_1, \ldots, X_n)$

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Upper bound for continuous operations

Definition (Continuous operations)

A limit of sets $\{A_n\}_{n \geqslant 1}$:

$$A = \lim_{n \to \infty} A_n \iff \begin{cases} x \in A & \text{if } x \text{ is in almost all } A_i\text{'s} \\ x \notin A & \text{if } x \text{ is in finitely many } A_i\text{'s} \end{cases}$$

An operation φ is continuous, if

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Theorem

If $L \subseteq \Omega^*$ is given by unique solution of a system with continuous (computable) operations, then L is recursive.

Equations with non-continuous operations: example

Example (RE sets by non-continuous operations)

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- $VALC(M)$—language of computations of TM.

$$\{C_M(w)\#w \mid w \in L(M)\}$$

- $w \in \Omega'^*$, $C_M(w), \# \in (\Omega \setminus \Omega')^*$
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- deleting $C_M(w)$ out of $VALC(M)$:

$$\pi_{\Omega'}(VALC(M)) = L(M)$$

Obvious upper bound

Set equation translates into formulas:

$$X_i = X_j + X_k \iff (\forall n)\big[n \in X_i \leftrightarrow (\exists n', n'')n = n' + n'' \wedge n' \in X_j \wedge n'' \in X_k\big]$$

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Unique solution $(S_1, \ldots, S_n)$:

$$\begin{aligned} \varphi(x) &= (\exists X_1) \ldots (\exists X_n) Eq(X_1, \ldots, X_n) \wedge x \in X_1 \quad (\Sigma^1_1) \\ \varphi'(x) &= (\forall X_1) \ldots (\forall X_n) Eq(X_1, \ldots, X_n) \rightarrow x \in X_1 \quad (\Pi^1_1) \end{aligned}$$

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\Delta^1_1 &= \Sigma^1_1 \cap \Pi^1_1 \quad (\text{Hyper-arithmetic sets})
\end{aligned}$$

Result

Theorem

$S \subseteq \mathbb{Z}$ is given by unique solution of a system with $\{\cup, +\}$ or $\{\cap, +\}$ and equations of the form $\varphi(X_1, \ldots, X_n) = \psi(X_1, \ldots, X_n)$

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Similar result for $\mathbb{N}$ with subtraction:

$$A \dot{-} B = \{a - b \mid a \in A, b \in B, a \geqslant b\}$$

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Example (Jeż, DLT 2007)

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X_1 &= (X_2 + X_2 \cap X_1 + X_3) \cup \{1\} \\
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Arithmetical Hierarchy

$AH = \bigcup_k \Sigma_k^0$, where

$S \in \Sigma_k^0 \iff S = \{w \mid \exists x_1 \forall x_2 \dots Q_k x_k \; R(w, x_1, \dots, x_k)\}$, recursive R

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An encoding

$$S = \{ (w)_7 \mid \exists x_1 \in \{3,6\}^* \, \forall x_2 \in \{3,6\}^* \dots Q_k x_k \in \{3,6\}^* \; (1 x_1 1 x_2 1 \dots x_k 1 w)_7 \in R\}$$

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Construction outline

- Recursive sets [Jeż, Okhotin ICALP 2008]—given by unique solutions over $\mathbb{N}$ with $\{\cap, +\}$ or $\{\cup, +\}$
- Enough to define quantifier operations

$$E(X) = \{(1w)_7 \mid \exists x : (1x1w)_7 \in X\}$$
$$A(X) = \{(1w)_7 \mid \forall x : (1x1w)_7 \in X\}$$

Existential quantifier

Theorem

For $S \subseteq (\{3,6\}^+ 1\Omega_7^*)_7$

$$\left[S \dot{-} (\{3,6\}^+ 0^*)_7\right] \cap (1\Omega_7^*)_7 = \{(1y)_7 \mid \exists x \in \{3,6\}^+ (x1y)_7 \in S\}$$

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- Goal:

$$\begin{array}{rcccccccccc} & x_1 & x_2 & \dots & x_\ell & 1 & y_1 & y_2 & \dots & y_{\ell'} \\ - & x_1 & x_2 & \dots & x_\ell & 0 & 0 & 0 & \dots & 0 \\ \hline & & & & & 1 & y_1 & y_2 & \dots & y_{\ell'} \end{array}$$

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- Reality:

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$$\Big((x1y)_7 \dot{-} (x'0\underbrace{0\dots0}_{?})_7\Big) \cap (1\Omega_7^*)_7$$

- filtered out:
  - $x \neq x'$
  - number of 0'es is wrong

Hyper-arithmetical sets

Definition (Effective σ union and intersection)

$f_1, f_2, \dots$ — enumeration of all partial recursive functions
τ_1, τ_2 — recursive functions (some assumptions),
$B_{\tau_1(k)} = \mathbb{N} \setminus \{k\}$, $C_{\tau_1(k)} = \{k\}$. If f_k is a total function, then

$$B_{\tau_2(k)} = \bigcup_{n \in \mathbb{N}} C_{f_k(n)}, \qquad C_{\tau_2(k)} = \bigcap_{n \in \mathbb{N}} B_{f_k(n)},$$

Hyper-arithmetical sets

Definition (Effective σ union and intersection)

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HA: smallest eff. σ-ring.

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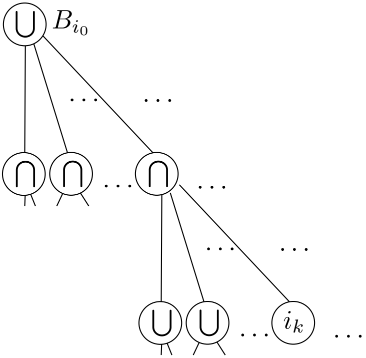
HA as trees

Imagine the dependencies on the tree.



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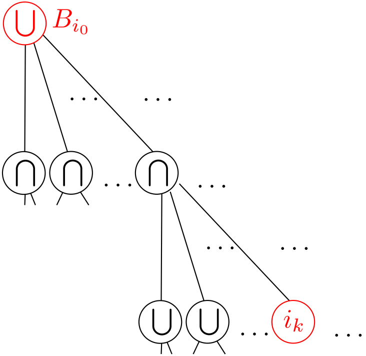

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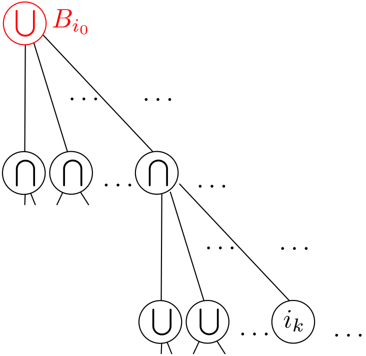

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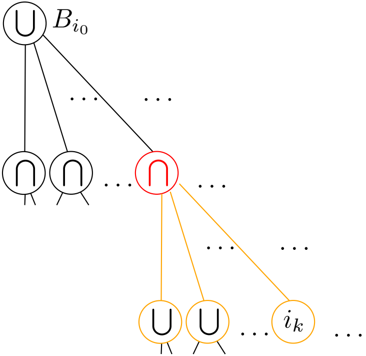

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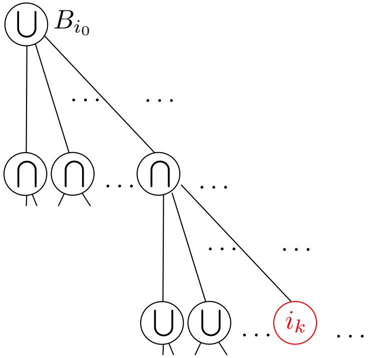

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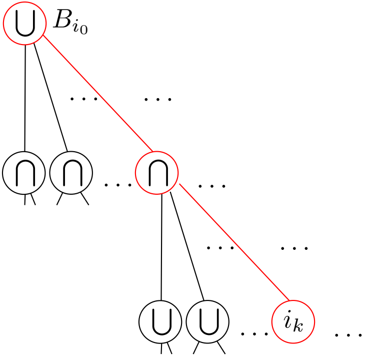

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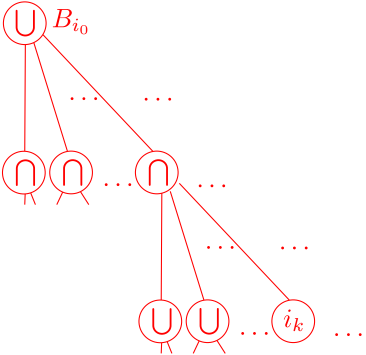

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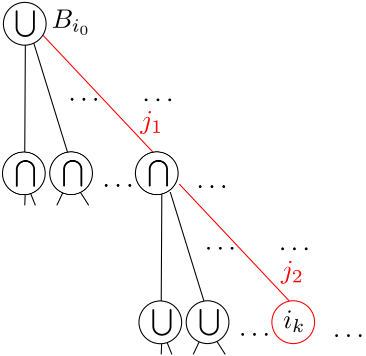

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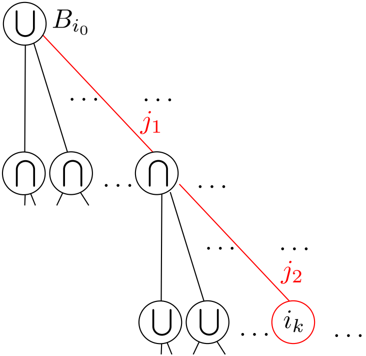

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Result[Jeż, Okhotin, STACS 2009]

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  - Using similar technique to equations over $\mathbb{N}$.