Uniwersytet Wrocławski

Word equations in nondeterministic linear space

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Given equation $U = V$, where $U, V \in (\Sigma \cup \mathcal{X})^*$.
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This is important

- unification
- word combinatorics
- helpful in equations in free group (and other)

Makanin '77 4NEXPTIME

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[. . .]

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Gutierrez '98 EXPSPACE

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Plandowski & Rytter '98 new approach — using compression

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This talk

Word Equations are in NLinSPACE

Main idea

- Recompression algorithm [J. 2013]
- Huffman coding of letters

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- Huffman coding of letters

The proof is more complex

- how letters depend on fragments of original equation
- special coding (so worse than Huffman)
- technically involved

Compression operations

Given a word w:

(Σ_ℓ, Σ_r) pair compression replace each $ab \in \Sigma_\ell\Sigma_r$ in w with fresh c_{ab}
(Σ_ℓ, Σ_r are disjoint)

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- We want to perform it on $S(U)$ and $S(V)$.
- Occurrence can be partially in the equation and in the variable.

Preliminaries: explicit word

Checking equality of two explicit words

Require: two words u, v to be tested for equality

```
while |u| > 1 or |v| > 1 do
    Σ ← letters in u, v
    perform Σ-block compression
    while some pair in Σ² was not considered do
        guess partition of Σ to (Σ_ℓ, Σ_r)
        perform (Σ_ℓ, Σ_r) pair compression
test equality
```

Checking equality of two explicit words

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1: **while** $|u| > 1$ or $|v| > 1$ **do**
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4: **while** some pair in Σ^2 was not considered **do**
5: guess partition of Σ to (Σ_ℓ, Σ_r)
6: perform (Σ_ℓ, Σ_r) pair compression
7: test equality

Phase: one iteration of the main loop.

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Shortening

Consider consecutive ab in u, v at the beginning of the phase

$a = b$ compressed as a block

$a \neq b$ considered and compressed, or
one of them was compressed earlier

Pair Compression on word equation

In a solution word $S(U)$ or $S(V)$:

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PairCompression(Σ_ℓ, Σ_r)

1: **for** $X \in \mathcal{X}$ **do**
2: let b: first letter of $S(X)$ ▷ Guess
3: **if** $b \in \Sigma_r$ **then**
4: replace each occurrence of X by bX ▷ Pop
5: **if** $S(X) = \epsilon$ **then** ▷ Guess
6: remove X from the equation
7: let a: last . . . ▷ symmetrically for the last letter and Σ_ℓ
8: perform pair compression on sides of the equation

Block Compression

BlockCompression

for $X \in \mathcal{X}$ **do**

  let $S(X) = a^\ell w b^r$ ▷ Guess

  replace X with $a^\ell X b^r$

  if $S(X) = \epsilon$ **then** ▷ Guess

    remove X from the equation

perform block compression on sides of the equation

The algorithm

Main algorithm

1: **while** sides of the equation are nontrivial **do**
2: $\Sigma \leftarrow$ letters in the equation
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Encoding

We use Huffman coding for letters. (Need to recalculate it.)
We use different encoding in the analysis.

We modify the equation, but think that we operate on $S(U) = S(V)$.
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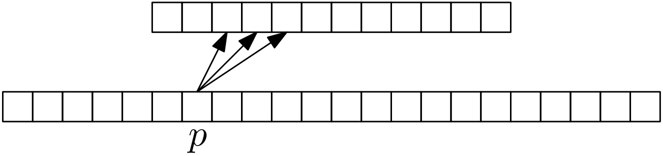
In NLinSPACE we can analyse only "good choices":
if we exceed the space then we reject.

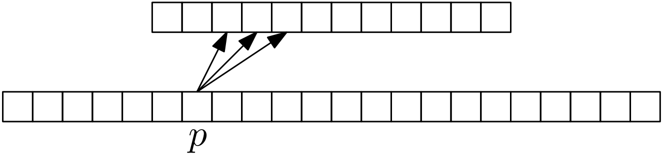
Dependency interval



Definition (Dependency interval)

An interval of positions in the input equation is called a dependency interval (depint).
We associate a depint to each symbol in the equation; $D = \mathrm{dep}(p)$.

Dependency interval



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Assigning depints

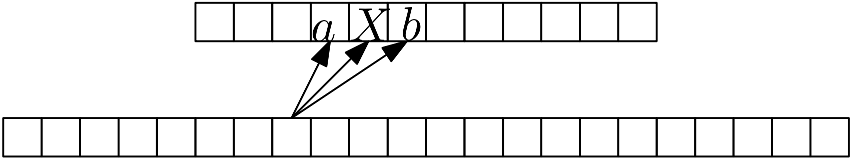
- Technical, operational manner.
- We expand the depints by taking unions with neighbouring ones.
- Popped letters have depints of their variables.
- Depints of letters introduced due to compression do not change.

Depints and encoding

Encoding

- letter at position $p \to UV[\mathrm{dep}(p)]$
- letters with this interval assigned are numbered $1, 2, \ldots, k$
- we assign to them codes $UV[D]\#1, UV[D]\#2, \ldots, UV[D]\#k$

Depints and encoding



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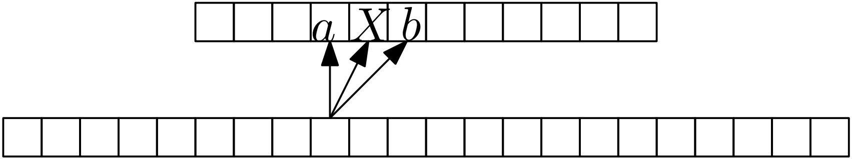

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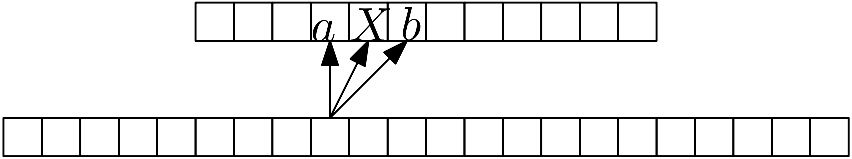

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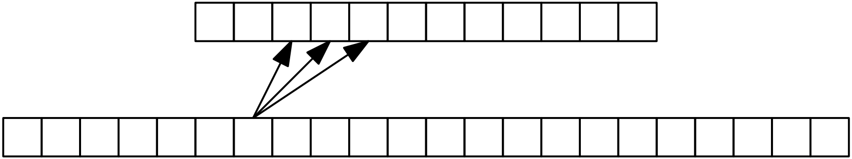
- formally not encoding: assigns different codes to the same letter
- never assigns the same code to different letters
- worse than Huffman coding; enough to estimate its bit-size

Depints size



Depints: positions to indices

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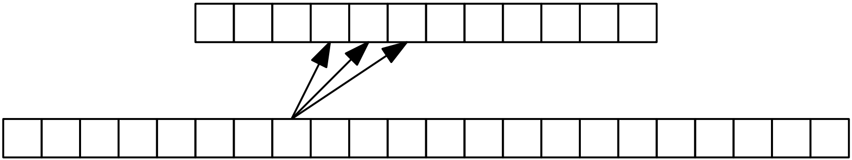

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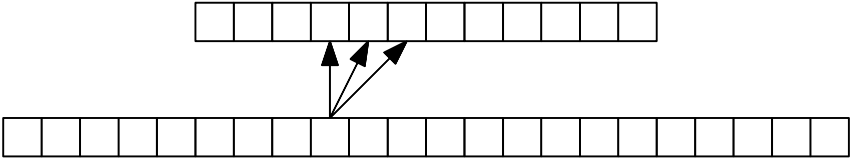

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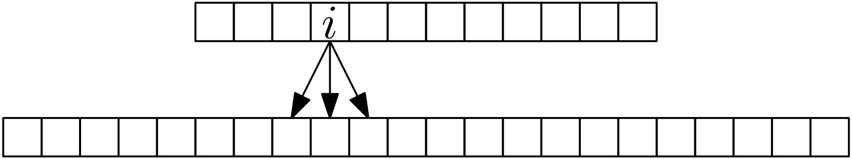

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Index to positions Pos(i)

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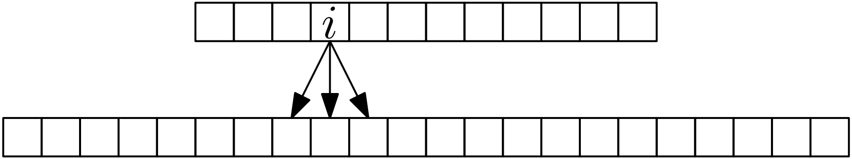

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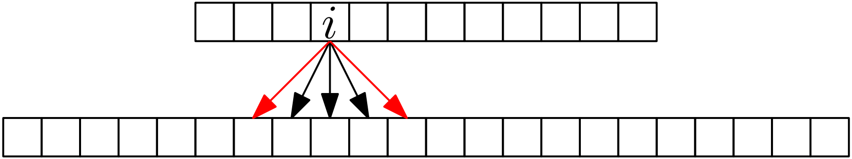

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Pos(i) grows: extending, popping letters

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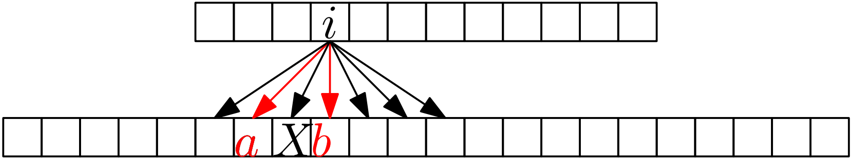

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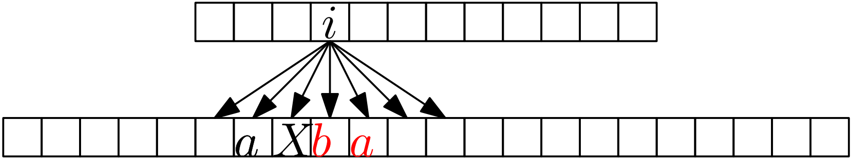

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Pos(i) shrinks: compressions

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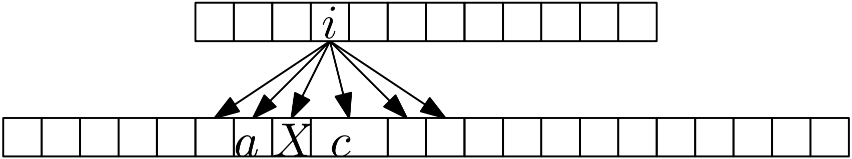

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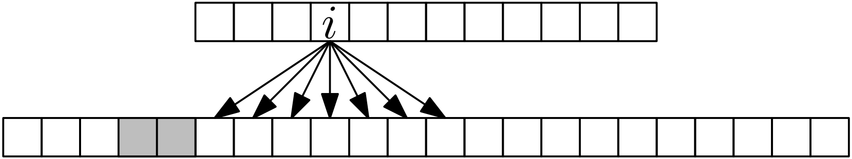

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Fresh letters block:

Letter to the left of Pos(i) is new — no extensions

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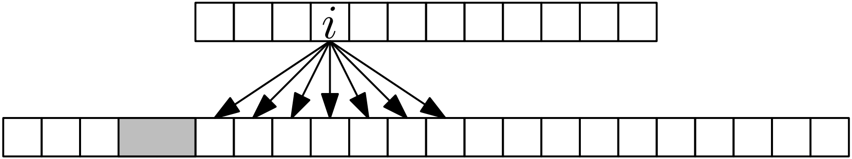

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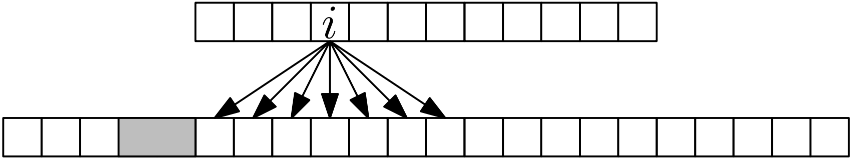

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Left letter in $S(X)$ is new — no popping

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- Our only choice that affects size is the partition.
- Choose the partitions to minimise bit size.
- If $\mathsf{Pos}(i) = \mathcal{O}(1)$ then everything works.

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Random partition to expectation

- Random compresses a pair with probability $1/4$.
- Each blocking is with probability $1/4$.
- Turn this into expectation: calculate what to minimise: length, frequency, new letters, number of occurrences, ...

$$\sum_{i \geq 0} \frac{1}{2^i} = 2$$

$$\sum_{i \geq 0} \frac{i^2 \log i}{2^i} = \mathcal{O}(1)$$

Other technicalities

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- need to change Huffman coding
- how to make block compression (no explicit numbers — known)
- what happens with the solution
- ending markers with special treatment
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Works for Huffman coding of the input.

Open questions

- Are word equations in NP?
- Can this be generalised to other equations? (constraints, involution, commutation)