Computational completeness of equations over sets of natural numbers

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Language equations

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Greatest solution: Σ^*.

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- Leiss (1995), Okhotin/Yakimova (2006), Jeż (2007), Jeż/Okhotin (2007–present): equations over $\{a\}$.

Computational completeness of language equations

- Language equations over Σ, with $|\Sigma| \geqslant 2$.

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✓ Remaking the argument for the unary case!

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Every context-free language over $\{a\}$ is regular.

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- The power of conjunctive grammars over $\{a\}$?

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Quadruple $G = (\Sigma, N, P, S)$, where...

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Conjunctive grammars (Okhotin, 2000) Rules of the form

$$A \to \alpha_1 \& \ldots \& \alpha_m$$

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  - Conjunctive grammar for $\{a^{4^n} \mid n \geqslant 0\}$.

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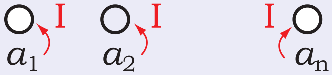

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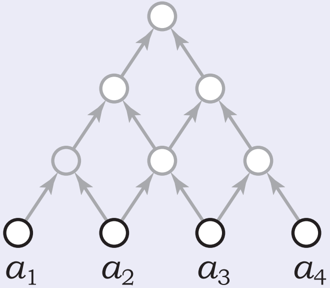

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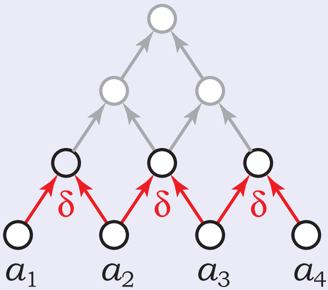

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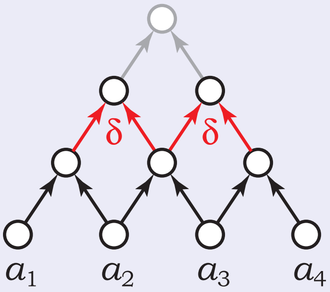

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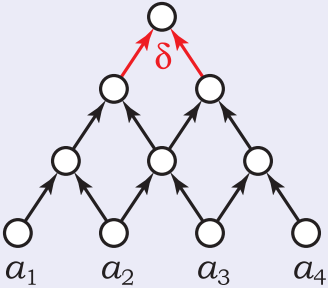

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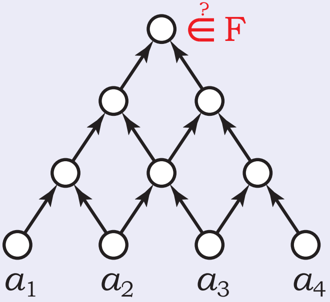

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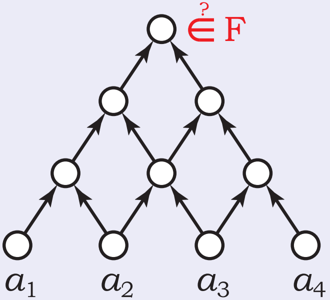

- Can recognize $\{wcw\}$, $\{a^n b^n c^n\}$, $\{a^n b^{2^n}\}$, VALC.

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Results for unresolved equations with $\{\cup, +\}$ or $\{\cap, +\}$

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Problem

Construct any simple system using $\{\cup, +\}$ with a non-periodic solution.