On the Two-Dimensional Cow Search Problem

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Abstract

We investigate the problem introduced by Baeza-Yates et al. [1]: given a plane and a horizontal or a vertical line in unknown location give a strategy to find this line. We use a competitive analysis to measure the performance of this strategy. This problem is one of the first generalisations of the cow search problem.
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1. Introduction

In their seminal paper Baeza-Yates et al. [1] introduced the problem of searching with uncertainty: in a general setting we are given some geometric space (for example a straight line) and we want to find its specific subspace (for example a point) while minimising the cost of this strategy (for example the length of the path to a point on the line). The example is usually decorated with a story of an imaginary cow starting from point 0 on the straight line and searching for a fence, which is known to exist somewhere on the line.

Our goal is to devise a strategy with small competitive ratio [2], ie. a strategy S is called R-*competitive* if the cost of S on any instance I is at most the optimum cost on I multiplied by R. The *competitive ratio* of S is the smallest R for which S is R-competitive.

Baeza-Yates et al. [1] have inspected several interesting cases of search problems, including searching on rays, planes, lines, stars, etc. The search for a point on the line made its way to the computer science folklore. Lower and upper bounds of 9 were given there and are one of the usual introductionary examples at the courses on on-line algorithms. Some of the problems stated in this paper and other, closely related, were afterwards considered by many researchers [6,4,7]. We refer the reader to [3] for a more modern and general reflection on the subject, or [5] for a more classic approach to search problems.

Among the problems stated [1] was an *Orthogonal Orientation Fence Arbitrarily Far Away 2-dimensional Cow Search Problem*, or in other words, a search for a horizontal or vertical line

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on the plane. Being more formal, we start at point $(0, 0)$ at the plane and we know that there is a distinguished line parallel to one of the axis on the plane. Our goal is to give a strategy (ie. path) for searching for such a line. The cost of the strategy on an instance is simply the length of the segment of the path from point $(0, 0)$ to the first intersection with the designated line. It is good to assume that the line cannot be too close to point $(0, 0)$, say has to be at distance at least 1—otherwise the optimal strategy may have arbitrarily small cost, thus causing the competitive ratio to be unbounded. We improve the upper bound of ≈ 13.02 [1] to ≈ 12.5406. To our best knowledge there were no successful attempts on finding any lower or upper bound on this problem. We note that if we restrict the strategies to logarithmic spirals then a lower bound greater than 12.6 can be shown [1].

2. Idea

In this section, we present an informal introduction and the idea behind the algorithm presented in the following section.

2.1. *Notation*

All the search strategies considered in this paper consist of piecewise straight segments and thus are described by a sequence of points. The walk in search of the line is simply a path joining the consecutive points.

Although this has no particular influence on the proof nor the construction, we usually think of such a path as a function of time. Thus the point at time t is described by a pair $(x(t), y(t))$. By $d(t)$ or d_t we denote the length of the path up to time t. Since we investigate mainly the projection of such walks into the vertical and horizontal lines, by $d_x(t)$ we denote the length of its projection on the line $y = 0$ and by $d_y(t)$ the length of its projection on the line $x = 0$.

Since in general the strategy can treat x and y coordinates differently we distinguish between the cases when the line is horizontal and vertical: we say that the line *is on* x or was *found on* x if the line is vertical and *is on* y or *found on* y otherwise.

2.2. *Toy example*

It is well known [1] that in case of searching on the line an optimal strategy consists of a sequence

$$1, -2, 4, -8, \dots \ .$$

In particular this strategy has a *constant expanding factor* property: the points in which the strategy changes direction form a geometric sequence. More general, in case of two dimensional search we say that a strategy has a constant expanding factor property if its projection on both x and on y have constant expanding factor property.

Consider a strategy for the two dimensional search

$$(1, 1), (-2, -2), (4, 4), \dots \ .$$

In the same way as in case of one dimensional search we can show that the competitive ratio is $9\sqrt{2}$: consider a projection of the motion into $x(t)$ and $y(t)$. Then when the line is found, wlog

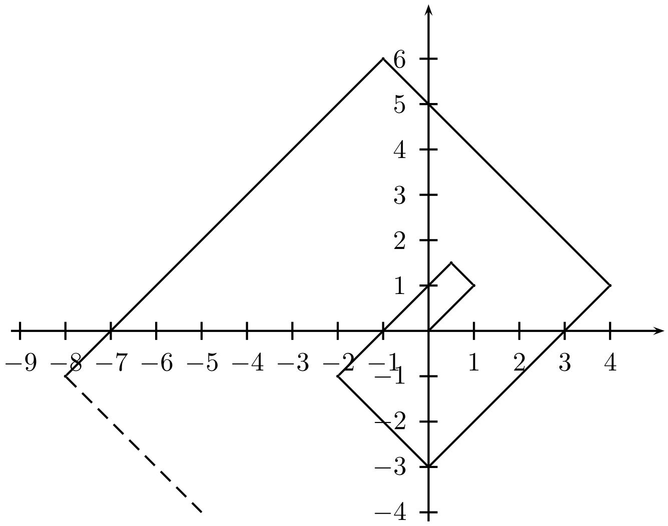

Fig. 1. A spiral strategy with competitive ratio $9\sqrt{2}$

we may assume that it was found on the x coordinate. As the length of the walk is $\sqrt{2}$ times longer than its projection, the claim follows.

2.3. *Spiral example*

The example from the previous subsection is hard to generalise. Here we present a slightly different one. Consider a strategy uniquely defined by:

$$x_{2i+1} = (-2)^i \ ,$$
$$y_{2i+2} = 1,5 \cdot (-2)^i \ .$$

and two properties
– each segment $(x_{2i}, y_{2i})(x_{2i+1}, y_{2i+1})$ is a straight line parallel to line $y = x$
– each segment $(x_{2i+1}, y_{2i+1})(x_{2i+2}, y_{2i+2})$ is a straight line parallel to line $y = -x$.
please cf. Fig. 1. The exact value of points x_{2i} and y_{2i+1} can be given by arithmetical expressions but they are quite complicated and thus we prefer not to give them explicitly. Clearly $d_x(t) = d_y(t)$. By the same reasoning as in the previous section, we can show that the competitive ratio of this strategy is $9\sqrt{2}$. Please note that the trajectory of this strategy is a discrete spiral and the arcs between two consecutive line segments are always of value $\frac{\pi}{2}$.

Our main result should be seen as a generalisation of the second example. We again use a path forming a discrete spiral, although a bit more sophisticated one.

2.4. *Different vertical and horizontal speeds*

Suppose, that at some point of time t we found the line on x. If $d_y(t) = d_x(t)$ then we basically expect $d(t) \geq \sqrt{2} \cdot d_x(t)$ and thus the competitive ratio to be at least $\sqrt{2} \cdot \frac{d_x(t)}{|x(t)|}$ and this has to be at least $9\sqrt{2}$ from time to time. Thus $9\sqrt{2}$ seems somehow unbeatable if $d_y(t) = d_x(t)$ for all t. On the other hand, if we manage to obtain $d_y(t) < d_x(t)$ then it is reasonable to assume that

$d(t) \leq \sqrt{2} \cdot d_x(t)$. A similar argument can be given for the y coordinate—as the line can be on y we have to proceed at the y coordinate. Hence the ratio $\frac{d_x(t)}{d_y(t)}$ has to oscillate and it is our task to find some characterisation of the times/places where it should increase and where it should decrease. In fact this is quite simple: by standard analysis one can show that any reasonable strategy for one dimensional search consists of alternating sweeps in the opposite directions, ie. the one-dimensional strategy is characterised by a sequence of points $x_1, x_2, x_3, \ldots$ such that x_i and x_{i+1} are of opposite signs and $x_{i+2} > x_i$. Suppose that the line was found when we go from x_i to x_{i+1} at point x'. Clearly it can happen only if $x' \in (x_{i-1}, x_{i+1}]$, since $[x_i, x_{i-1}]$ is already explored. Consider the competitive ratio as a function of x'. Then it can be shown, that the competitive ratio is highest, when $x' = x_{i-1}$ and that it decreases as x' increases. Thus it is reasonable to construct a sequence, in which $d_x(t)$ will increase faster than $d_y(t)$ when going from x_i to x_{i-1} and then will increase slower when going from x_{i-1} to x_{i+1}. The same should happen for the y coordinate.

3. Upper bound

In this section we formalise the ideas above

3.1. *Construction*

The strategy is described by a sequence of points (x_i, y_i). We assume, that the projections $x(t)$ and $y(t)$ have a constant factor expanding property with coefficient α. The intuition described in the previous subsection suggests that the following equations should be satisfied (we replace x_i from that discussion by x_{2i}):

$$\begin{aligned} x_{2k+1} &= -\alpha x_{2k-1} \quad , \\ x_{2k} &= x_{2k-3} \quad , \\ y_{2k+2} &= -\alpha y_{2k} \quad , \\ y_{2k+1} &= y_{2k-2} \quad . \end{aligned}$$

We have to choose starting values, wlog take $x_1 = 1$ and $y_2 = \beta$. There is ambiguity in defining x_2 and y_3, we have chosen those values so that the formulas are uniform for all points. It is easy to see that the solution of those equations is

$$\begin{aligned} x_{2k} &= (-1)^k \alpha^{k-2} \quad , \\ x_{2k+1} &= (-1)^k \alpha^k \quad , \\ y_{2k} &= (-1)^{k+1} \beta \alpha^{k-1} \quad , \\ y_{2k+1} &= (-1)^k \beta \alpha^{k-2} \quad . \end{aligned}$$

An example of such a strategy for some arbitrary α and β is given on Fig. 2.

3.2. *Calculation*

We now calculate the distances and competitive ratio in terms of α and β. To this end for each pair of points $(x_i, y_i), (x_{i-1}, y_{i-1})$ we calculate the distance $|x_i - x_{i-1}|$, analogously $|y_i - y_{i-1}|$

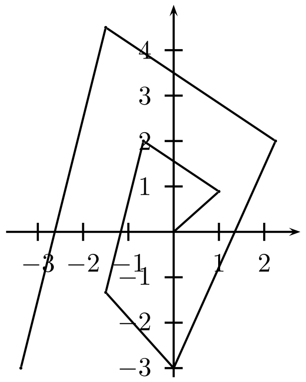

Fig. 2. An example of a spiral strategy for $\alpha = 1,5$ and $\beta = 2$

and then the distance between those two consecutive points. There are cases depending on the parity of i. To shorten the notation let d_i be the distance between (x_i, y_i) and (x_{i-1}, y_{i-1}):

$$
\begin{aligned}
|x_{2k} - x_{2k-1}| &= \alpha^{k-2} + \alpha^{k-1} = \alpha^{k-2}(\alpha + 1) \ , \\
|x_{2k+1} - x_{2k}| &= \alpha^{k} + \alpha^{k-2} = \alpha^{k-2}(\alpha^2 - 1) \ , \\
|y_{2k} - y_{2k-1}| &= \beta\alpha^{k-1} - \beta\alpha^{k-3} = \beta\alpha^{k-3}(\alpha^2 - 1) \ , \\
|y_{2k+1} - y_{2k}| &= \beta\alpha^{k-1} + \beta\alpha^{k-2} = \beta\alpha^{k-2}(\alpha + 1) \ ,
\end{aligned}
$$

and hence

$$
\begin{aligned}
d_{2k} &:= d\Big((x_{2k}, y_{2k}), (x_{2k-1}, y_{2k-1})\Big) = \alpha^{k-3}(\alpha+1)\sqrt{\alpha^2 + \beta^2(\alpha-1)^2} \ , \\
d_{2k+1} &:= d\Big((x_{2k+1}, y_{2k+1}), (x_{2k}, y_{2k})\Big) = \alpha^{k-2}(\alpha+1)\sqrt{\beta^2 + (\alpha-1)^2} \ .
\end{aligned}
$$

Consider d_x moving from x_{2k+3} to x_{2k+5}. Since the segment $[x_{2k+3,x_{2k+4}}]$ is explored, then a line can be found on x only when going from x_{2k+4} to x_{2k+5}. Suppose that the line was found when a $\xi \in [0,1]$ part of this segment was traversed. In other words are at point $x(t) = x_{2k+4} + \xi(x_{2k+5} - x_{2k+4})$. Then $d(t) = \sum_{i=0}^{k+2} d_{2i} + \sum_{i=0}^{k+1} d_{2i+1} + \xi d_{2i+5}$ and hence the ratio is

$$
\frac{\sum_{i=0}^{k+2} d_{2i} + \sum_{i=0}^{k+1} d_{2i+1} + \xi d_{2i+5}}{x_{2k+4} + \xi(x_{2k+5} - x_{2k+4})} \leq \max\left(\frac{\sum_{i=0}^{k+2} d_{2i} + \sum_{i=0}^{k+1} d_{2i+1}}{x_{2k+4}}, \frac{d_{2i+5}}{(x_{2k+5} - x_{2k+4})}\right) \ .
$$

Similarly, when the line is found on $y(t) = y_{2k+3} + \xi(y_{2k+4} - y_{2k+4})$:

$$
\frac{\sum_{i=0}^{k+1} d_{2i} + \sum_{i=0}^{k+1} d_{2i+1} + \xi d_{2i+5}}{y_{2k+3} + \xi(y_{2k+4} - y_{2k+3})} \leq \max\left(\frac{\sum_{i=0}^{k+1} d_{2i} + \sum_{i=0}^{k+1} d_{2i+1}}{y_{2k+3}}, \frac{d_{2i+5}}{(y_{2k+4} - x_{2k+3})}\right) \ .
$$

Both of the second terms are small:

$$\frac{d_{2i+5}}{(x_{2k+5}-x_{2k+4})} = \frac{\alpha^k(\alpha+1)\sqrt{\beta^2+(\alpha-1)^2}}{\alpha^k(\alpha^2-1)} = \frac{\sqrt{\beta^2+(\alpha-1)^2}}{\alpha-1} \ , \tag{1}$$

$$\frac{d_{2i+4}}{(y_{2k+4}-x_{2k+3})} = \frac{\alpha^{k-1}(\alpha+1)\sqrt{\alpha^2+\beta^2(\alpha-1)^2}}{\beta\alpha^{k-1}(\alpha^2-1)} = \frac{\sqrt{(\frac{\alpha}{\beta})^2+(\alpha-1)^2}}{(\alpha-1)} \ . \tag{2}$$

We later show that they are small for the optimal value of parameters. For the time being let us consider $\xi = 0$. Thus we need only to upper bound

$$\frac{\sum_{i=0}^{k+2} d_{2i} + \sum_{i=0}^{k+1} d_{2i+1}}{|x_{2k+4}|} \quad \text{and} \quad \frac{\sum_{i=0}^{k+1} d_{2i} + \sum_{i=0}^{k+1} d_{2i+1}}{|y_{2k+3}|} \ .$$

We calculate

$$\begin{aligned}
\sum_{i=0}^{k+2} d_{2i} &= \sum_{i=0}^{k+2} \alpha^{i-3}(\alpha+1)\sqrt{\alpha^2+\beta^2(\alpha-1)^2} \\
&\le \frac{\alpha^k}{\alpha-1}(\alpha+1)\sqrt{\alpha^2+\beta^2(\alpha-1)^2} \ , \\
\sum_{i=0}^{k+1} d_{2i+1} &= \sum_{i=0}^{k+1} \alpha^{i-2}(\alpha+1)\sqrt{\beta^2+(\alpha-1)^2} \\
&\le \frac{\alpha^k}{\alpha-1}(\alpha+1)\sqrt{\beta^2+(\alpha-1)^2} \ , \\
\sum_{i=0}^{k+1} d_{2i} &\le \frac{\alpha^{k-1}}{\alpha-1}(\alpha+1)\sqrt{\beta^2+(\alpha-1)^2} \ .
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\sum_{i=0}^{k+2} d_{2i} + \sum_{i=0}^{k+1} d_{2i+1}}{|x_{2k+4}|} &\le \frac{1}{\alpha-1}\Big((\alpha+1)\sqrt{\alpha^2+\beta^2(\alpha-1)^2} + (\alpha+1)\sqrt{\beta^2+(\alpha-1)^2}\Big) \\
&= \frac{\alpha+1}{\alpha-1}\Big(\sqrt{\alpha^2+\beta^2(\alpha-1)^2} + \sqrt{\beta^2+(\alpha-1)^2}\Big) \ , &(3) \\
\frac{\sum_{i=0}^{k+1} d_{2i} + \sum_{i=0}^{k+1} d_{2i+1}}{|y_{2k+3}|} &\le \frac{1}{\beta(\alpha-1)}\Big((\alpha+1)\sqrt{\alpha^2+\beta^2(\alpha-1)^2} + \alpha(\alpha+1)\sqrt{\beta^2+(\alpha-1)^2}\Big) \\
&= \frac{\alpha+1}{\beta(\alpha-1)}\Big(\sqrt{\alpha^2+\beta^2(\alpha-1)^2} + \alpha\sqrt{\beta^2+(\alpha-1)^2}\Big) \ . &(4)
\end{aligned}$$

Fix α. Then (3) increases with β and (4) decreases with β. It is easy to see that they are equal for $\beta = \sqrt{\alpha}$. Substituting it gives:

$$\begin{aligned}
&\frac{\alpha+1}{\sqrt{\alpha}(\alpha-1)}\Big(\sqrt{\alpha^2+\alpha(\alpha-1)^2} + \alpha\sqrt{\alpha+(\alpha-1)^2}\Big) = \\
&= \frac{\alpha+1}{(\alpha-1)}\Big(\sqrt{\alpha+(\alpha-1)^2} + \sqrt{\alpha}\sqrt{\alpha+(\alpha-1)^2}\Big) = \\
&= \frac{\alpha+1}{\sqrt{\alpha}-1}\sqrt{\alpha+(\alpha-1)^2} = \frac{\alpha+1}{\sqrt{\alpha}-1}\sqrt{\alpha^2-\alpha+1} \ .
\end{aligned}$$

Let $f(\alpha) = \frac{\alpha+1}{\sqrt{\alpha}-1}\sqrt{\alpha^2-\alpha+1}$. We calculate the derivative:

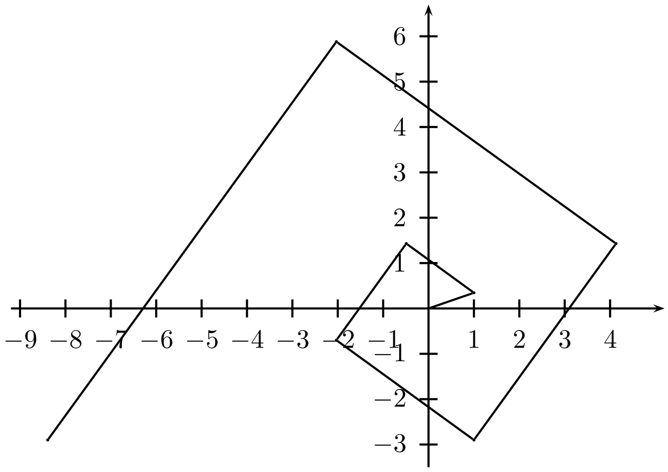

Fig. 3. The starting fragment of spiral strategy for optimal value of parameters.

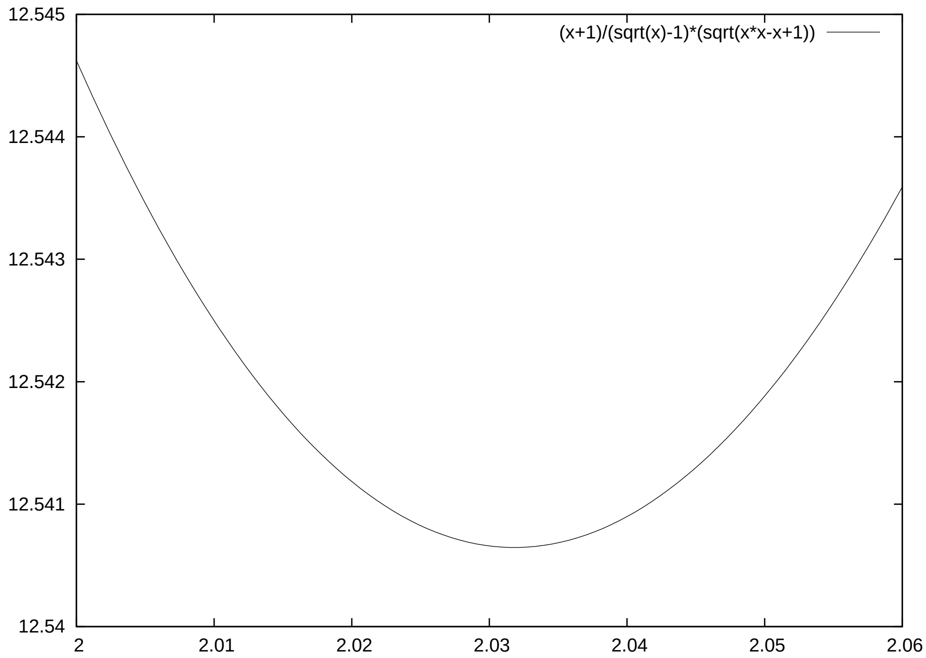

Fig. 4. A plot of $f(\alpha)$ for α around the point 2.03

$$\frac{(\alpha-2\sqrt{\alpha}-1)(\alpha^2-\alpha+1)+(\alpha+1)(2\alpha-1)\sqrt{\alpha}(\sqrt{\alpha}-1)}{2\sqrt{\alpha}(\sqrt{\alpha}-1)^2\sqrt{\alpha^2-\alpha+1}}\ .$$

We look for zeroes of the derivative. The numerical calculation shows that it is at point ≈ 2.03183 (surprisingly close to 2, but different). And therefore the value of f at this point is: $f(2.03183) \approx 12.5406$. A plot of values of f around the point 2.03 is given in Fig. 3.2 — it graphically justifies our choice of α. An illustration of the spiral with this parameter is given in Fig. 3.

We need only to check that (1) and (2) are small for those values of parameters:

$$\frac{\sqrt{\beta^2 + (\alpha-1)^2}}{\alpha - 1} = \frac{\sqrt{(\frac{\alpha}{\beta})^2 + (\alpha-1)^2}}{(\alpha-1)} = \frac{\sqrt{\alpha^2 - \alpha + 1}}{\alpha - 1} < 1,07$$

Thus the largest value of the competitive ratio of the strategy for those parameters is for $\xi = 0$ and its competitive ratio is at about $\approx 12,5406$.

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