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ON EQUATIONS OVER SETS OF INTEGERS

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Abstract. Systems of equations with sets of integers as unknowns are considered. It is shown that the class of sets representable by unique solutions of equations using the operations of union and addition $S+T = \{m+n \mid m \in S, n \in T\}$ and with ultimately periodic constants is exactly the class of hyper-arithmetical sets. Equations using addition only can represent every hyper-arithmetical set under a simple encoding. All hyper-arithmetical sets can also be represented by equations over sets of natural numbers equipped with union, addition and subtraction $S \dot{-} T = \{m-n \mid m \in S, n \in T, m \geqslant n\}$. Testing whether a given system has a solution is Σ^1_1-complete for each model. These results, in particular, settle the expressive power of the most general types of language equations, as well as equations over subsets of free groups.

1. Introduction

Language equations are equations with formal languages as unknowns. The simplest such equations are the context-free grammars [4], as well as their generalization, the conjunctive grammars [15]. Many other types of language equations have been studied in the recent years, see a survey by Kunc [11]. One of the research directions explores connection between language equations and computability. In particular, for equations with concatenation and Boolean operations it was shown by Okhotin [19, 17] that the class of languages representable by their unique (least, greatest) solutions is exactly the class of recursive (r.e., co-r.e.) sets. A computationally universal equation of the simplest form was constructed by Kunc [10], who proved that the greatest solution of the equation $XL = LX$, where $L \subseteq \{a,b\}^*$ is a finite constant language, may be co-r.e.-complete.

A seemingly trivial case of language equations over a *unary alphabet* $\Sigma = \{a\}$ has recently been studied. Strings over such an alphabet may be regarded as natural numbers,

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and languages accordingly become sets of numbers. As established by the authors [8], these equations are as powerful as language equations over a general alphabet: a set of natural numbers is representable by a unique solution of a system with union and elementwise addition iff it is recursive. Furthermore, even without the union operation these equations remain almost as powerful [9]: for every recursive set $S \subseteq \mathbb{N}$, its encoding $\sigma(S) \subseteq \mathbb{N}$ satisfying $S = \{n \mid 16n + 13 \in \sigma(S)\}$ can be represented by a unique solution of a system using addition only, as well as ultimately periodic constants. At the same time, as shown by Lehtinen and Okhotin [12], some recursive sets are not representable without an encoding.

Equations over sets of numbers are, on one hand, interesting on their own as a basic mathematical object. On the other hand, these equations form a very special case of language equations with concatenation and Boolean operations, which turned out to be as hard as the general case, and this is essential for understanding language equations. However, it must be noted that these cases do not exhaust all possible language equations. The recursive upper bound on unique solutions is applicable only to equations with *continuous* operations on languages, and using the simplest non-continuous operations, such as homomorphisms or quotient [18], leads out of the class of recursive languages. In particular, quotient with regular constants allows representing all sets in the arithmetical hierarchy [18].

The task is to find a natural limit of the expressive power of language equations, which would not assume continuity of operations. As long as operations on languages are expressible in first-order arithmetic (which is true for every common operation), it is not hard to see that unique solutions of equations with these operations always belong to the family of *hyper-arithmetical sets* [14, 20, 21]. This paper shows that this obvious upper bound is in fact reached already in the case of a unary alphabet.

To demonstrate this, two abstract models dealing with sets of numbers shall be introduced. The first model are equations over sets of natural numbers with addition $S + T = \{m + n \mid m \in S,\ n \in T\}$ and subtraction $S \dot{-} T = \{m - n \mid m \in S,\ n \in T,\ m \geqslant n\}$ (which corresponds to quotient of unary languages), as well as set-theoretic union. The other model has sets of integers, including negative numbers, as unknowns, and the allowed operations are addition and union. The main result of this paper is that unique solutions of systems of either kind can represent every *hyper-arithmetical* set of numbers.

The base of the construction is the authors' earlier result [8] on representing every recursive set by equations over sets of natural numbers with union and addition. In Section 2, this result is adapted to the new models introduced in this paper. The next task is representing every set in the arithmetical hierarchy, which is achieved in Section 3 by simulating existential and universal quantifiers over a recursive set. These arithmetical sets are then used in Section 4 as constants for the constuction of equations representing hyper-arithmetical sets. Finally, equations over sets of integers with addition only, using ultimately periodic constant sets.

This result brings to mind a study by Robinson [20], who considered equations, in which the constants and the unknowns are functions from $\mathbb{N}$ to $\mathbb{N}$, and the only operation is superposition, and proved that a function is representable by a unique solution of such an equation iff it is hyper-arithmetical. Though these equations deal with objects different from sets of numbers, there is one essential thing in common: in both results, unique solutions of equations over second-order arithmetical objects represent hyper-arithmetical sets.

Some more related work can be mentioned. Halpern [5] studied the decision problem of whether a formula of Presburger arithmetic with set variables is true for all values of these set variables, and showed that it is Π^1_1-complete. The equations studied in this paper can be regarded as a small fragment of Presburger arithmetic with set variables.

Another relevant model are languages over free groups, which have been investigated, in particular, by Anisimov [3] and by d'Alessandro and Sakarovitch [2]. Equations over sets of integers are essentially equations for languages over a monogenic free group.

An important special case of equations over sets of numbers are *expressions* and *circuits* over sets of numbers, which are equations without iterated dependencies. Expressions and circuits over sets of natural numbers were studied by McKenzie and Wagner [13], and a variant of these models defined over sets of integers was investigated by Travers [22].

2. Equations and their basic expressive power

The subject of this paper are systems of equations of the form

$$\begin{cases} \varphi_1(X_1,\ldots,X_n) &= \psi_1(X_1,\ldots,X_n) \\ &\vdots \\ \varphi_m(X_1,\ldots,X_n) &= \psi_m(X_1,\ldots,X_n) \end{cases}$$

where $X_i \subseteq \mathbb{Z}$ are unknown sets of integers, and the expressions φ_i and ψ_i use such operations as union, intersection, complementation, as well as the main arithmetical operation of elementwise addition of sets, defined as $S+T = \{m+n \mid m \in S,\ n \in T\}$. Subtraction $S-T = \{m-n \mid m \in S,\ n \in T\}$ shall be occasionally used. The constant sets contained in a system sometimes will be singletons only, sometimes any ultimately periodic constants will be allowed (a set of integers $S \subseteq \mathbb{Z}$ is *ultimately periodic* if there exist numbers $d \geqslant 0$ and $p \geqslant 1$, such that $n \in S$ iff $n+p \in S$ for all n with $|n| \geqslant d$), and in some cases the constants will be drawn from wider classes of sets, such as all recursive sets. Systems over sets of natural numbers shall have subsets of $\mathbb{N}$ both as unknowns and as constant languages; whenever subtraction is used in such equations, it will be used in the form $S \dot{-} T = (S-T) \cap \mathbb{N}$.

Consider systems with a unique solution. Every such system can be regarded as a specification of a set, and for every type of systems there is a natural question of what kind of sets can be represented by unique solutions of these systems. For equations over sets of natural numbers, these are the recursive sets:

Proposition 1 (Jeż, Okhotin [8, Thm. 4])**.** *The family of sets of natural numbers representable by unique solutions of systems of equations of the form $\varphi_i(X_1,\ldots,X_n) = \psi_i(X_1,\ldots,X_n)$ with union and addition, is exactly the family of recursive sets.*

Turning to the more general cases of equations over sets of integers and of equations over sets of natural numbers with subtraction, an upper bound on their expressive power can be obtained by reformulating a given system in the notation of first-order arithmetic.

Proposition 2. *For every system of equations in variables $X_1,\ldots X_n$ using operations expressible in first-order arithmetic there exists an arithmetical formula $Eq(X_1,\ldots,X_n)$, where $X_1,\ldots,X_n$ are free second-order variables, such that $Eq(S_1,\ldots,S_n)$ is true iff $X_i = S_i$ is a solution of the the system.*

Constructing this formula is only a matter of reformulation. As an example, an equation $X_i = X_j + X_k$ is represented by $(\forall n)\big[n \in X_i \leftrightarrow (\exists n')(\exists n'') n = n'+n'' \wedge n' \in X_j \wedge n'' \in X_k\big]$.

Now consider the following formulae of second-order arithmetic:

$$\varphi(x) = (\exists X_1) \dots (\exists X_n) Eq(X_1, \dots, X_n) \wedge x \in X_1$$
$$\varphi'(x) = (\forall X_1) \dots (\forall X_n) Eq(X_1, \dots, X_n) \wedge x \in X_1$$

The formula $\varphi(x)$ represents the membership of x in *any* solution of the system, while $\varphi'(x)$ states that *every* solution of the system contains x. Since, by assumption, the system has a unique solution, these two formulae are equivalent and each of them specifies the first component of this solution. Furthermore, φ and φ' belong to the classes Σ^1_1 and Π^1_1, respectively, and accordingly the solution belongs to the class $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$, known as the class of *hyper-arithmetical sets* [14, 21].

Proposition 3. *For every system of equations in variables $X_1, \dots X_n$ using operations expressible in first-order arithmetic that has a unique solution $X_i = S_i$, the sets S_i are hyper-arithmetical.*

Though this looks like a very rough upper bound, this paper actually establishes the converse, that is, that every hyper-arithmetical set is representable by a unique solution of such equations. The result shall apply to equations of two kinds: over sets of integers with union and addition, and over sets of natural numbers with union, addition and subtraction. In order to establish the properties of both families of equations within a single construction, the next lemma introduces a general form of systems that can be converted to either of the target types of systems:

Lemma 1. *Consider any system of equations $\varphi(X_1, \dots, X_m) = \psi(X_1, \dots, X_m)$ and inequalities $\varphi(X_1, \dots, X_m) \subseteq \psi(X_1, \dots, X_m)$ over sets of natural numbers that uses the following operations: union; addition of a recursive constant; subtraction of a recursive constant; intersection with a recursive constant. Assume that the system has a unique solution $X_i = S_i$. Then there exist:*

1. *a system of equations over sets of natural numbers in variables $X_1, \dots, X_m, Y_1, \dots, Y_{m'}$ using the operations of addition, subtraction and union and singleton constants which has a unique solution with $X_i = S_i$;*
2. *a system of equations over sets of integers in variables $X_1, \dots, X_m, Y_1, \dots, Y_{m'}$ using the operations of addition and union, singleton constants and the constants $\mathbb{N}$ and $-\mathbb{N}$, which has a unique solution with $X_i = S_i$.*

Inequalities $\varphi \subseteq \psi$ can be simulated by equations $\varphi \cup \psi = \psi$. For equations over sets of natural numbers, each recursive constant is represented according to Proposition 1, and this is enough to implement each addition or subtraction of a recursive constant by a large subsystem using only singleton constants. In order to obtain a system over sets of integers, a straightforward adaptation of Proposition 1 is needed:

Lemma 1.1. *For every recursive set $S \subseteq \mathbb{N}$ there exists a system of equations over sets of integers in variables $X_1, \dots, X_n$ using union, addition, singleton constants and constant $\mathbb{N}$, such that the system has a unique solution with $X_1 = S$.*

This is essentially the system given by Proposition 1, with additional equations $X_i \subseteq \mathbb{N}$.

Now a difference $X \dot{-} R$ for a regular constant $R \subseteq \mathbb{N}$ shall be represented as $(X + (-R)) \cap \mathbb{N}$, where the set $-R$ is represented by taking a system for R and applying the following transformation

Lemma 1.2 (Representing sets of opposite numbers). *Consider a system of equations over sets of integers, in variables $X_1, \ldots, X_n$, using the operations of union and addition, and any constant sets, which has a unique solution $X_i = S_i$. Then the same system, with each constant $C \subseteq \mathbb{Z}$ replaced by the set of opposite numbers $-C$, has the unique solution $X_i = -S_i$.*

The last step in the proof of Lemma 1 is eliminating intersection with recursive constants. This is done as follows:

Lemma 1.3 (Intersection with constants). *Let $R \subseteq \mathbb{N}$ be a recursive set. Then there exists a system of equations over sets of natural numbers using union, addition and singleton constants, which has variables $X, Y, Y', Z_1, \ldots, Z_m$, such that the set of solutions of this system is*

$$\bigl\{\, (X = S,\ Y = S \cap R,\ Y' = S \cap \overline{R},\ Z_i = S_i) \bigm| S \subseteq \mathbb{N} \,\bigr\},$$

where $S_1, \ldots, S_m$ are some fixed sets.

In plain words, the constructed system works as if an equation $Y = X \cap R$ (and also as another equation $Y' = X \cap \overline{R}$, which may be ignored).

This completes the transformations needed for Lemma 1.

The next lemma shows that a set of integers (both positive and negative) can be represented by first representing its positive and negative subsets individually.

Lemma 2 (Assembling positive and negative subsets). *If sets $S \cap \mathbb{N}$ and $(-S) \cap \mathbb{N}$ are representable by unique solutions of equations over sets of integers using union, addition, and ultimately periodic constants. Then S is representable by equations over integers using only union, addition and ultimately periodic constants.*

3. Representing the arithmetical hierarchy

Each arithmetical set can be represented by a recursive relation with a quantifier prefix, and arithmetical sets form the *arithmetical hierarchy* based on the number of quantifier alternations in such a formula. The bottom of the hierarchy are the recursive sets, and every next level is comprised of two classes, Σ^0_k or Π^0_k, which correspond to the cases of the first quantifier's being existential or universal. For every $k \geqslant 1$, a set is in Σ^0_k if it can be represented as

$$\{w \mid \exists x_1 \forall x_2 \ldots Q_k x_k\ R(w, x_1, \ldots, x_k)\}$$

for some recursive relation R, where $Q_k = \forall$ if k is even and $Q_k = \exists$ if k is odd. A set is in Π^0_k if it admits a similar representation with the quantifier prefix $\forall x_1 \exists x_2 \ldots Q_k x_k$. It is easy to see that $\Pi^0_k = \{L \mid \overline{L} \in \Sigma^0_k\}$. The sets Σ^0_1 and Π^0_1 are the recursively enumerable sets and their complements, respectively. The arithmetical hierarchy is known to be strict: $\Sigma^0_k \subset \Sigma^0_{k+1}$ and $\Pi^0_k \subset \Pi^0_{k+1}$ for every $k \geqslant 0$. Furthermore, for every $k \geqslant 1$ the inclusion $\Sigma^0_k \cup \Pi^0_k \subset \Sigma^0_{k+1} \cap \Pi^0_{k+1}$ is proper, i.e., there is a gap between the k-th and $(k+1)$-th level.

For this paper, the definition of arithmetical sets shall be arithmetized in base-7 notation[1] as follows: a set $S \subseteq \mathbb{N}$ is in Σ^0_k if it is representable as

$$S = \{\, (w)_7 \mid \exists x_1 \in \{\mathtt{3},\mathtt{6}\}^* \forall x_2 \in \{\mathtt{3},\mathtt{6}\}^* \ldots Q_k x_k \in \{\mathtt{3},\mathtt{6}\}^* (\mathtt{1}x_1\mathtt{1}y_1\mathtt{1} \ldots x_k\mathtt{1}y_k\mathtt{1}w)_7 \in R\},$$

[1]Base 7 is just the smallest base, for which the technical details of the below constructions could be conveniently implemented.

for some recursive set $R \subseteq \mathbb{N}$, where $(w)_7$ for $w \in \{\mathtt{0}, \mathtt{1}, \ldots, \mathtt{6}\}^*$ denotes the natural number with base-7 notation w. The strings $x_i \in \{\mathtt{3}, \mathtt{6}\}$ represent *binary* notation of some numbers, where 3 stands for zero and 6 stands for one. The notation $(x)_2$ for $x \in \{\mathtt{3}, \mathtt{6}\}^*$ shall be used to denote the number represented by this encoding. The digits 1 act as separators. Throughout this paper, the set of base-7 digits $\{\mathtt{0}, \mathtt{1}, \ldots, \mathtt{6}\}$ shall be denoted by Σ_7.

In general, the construction of a system of equations representing the set S begins with representing R, and proceeds with evaluating the quantifiers, eliminating the prefixes $\mathtt{1}x_1$, $\mathtt{1}x_2$, and so on until $\mathtt{1}x_k$. In the end, all numbers $(\mathtt{1}w)_7$ with $(w)_7 \in S$ will be produced. These manipulations can be expressed in terms of the following three functions:

$$\begin{aligned} \mathit{Remove}_{\mathtt{1}}(X) &= \{(w)_7 \mid (\mathtt{1}w)_7 \in X\}, \\ E(X) &= \{(\mathtt{1}w)_7 \mid \exists x \in \{\mathtt{3}, \mathtt{6}\}^* : (x\mathtt{1}w)_7 \in X\}, \\ A(X) &= \{(\mathtt{1}w)_7 \mid \forall x \in \{\mathtt{3}, \mathtt{6}\}^* : (x\mathtt{1}w)_7 \in X\}. \end{aligned}$$

The expression converting numbers of the form $(\mathtt{1}w)_7$ to $(w)_7$ is constructed as follows:

Lemma 3 (Removing leading digit 1)**.** *The value of the expression*

$$(X - \{1\} \cap \{0\}) \cup \bigcup_{i \in \Sigma_7 \setminus \{\mathtt{0}\}} \bigcup_{t \in \{0,1\}} (X \cap (\mathtt{1}i\Sigma_7^t(\Sigma_7^2)^*)_7) - (\mathtt{10}^*)_7 \cap (i\Sigma_7^t(\Sigma_7^2)^*)_7 \qquad (3.1)$$

on any $S \subseteq (\mathtt{1}(\Sigma_7^* \setminus \mathtt{0}\Sigma_7^*))_7$ *is* $\{(w)_7 \mid (\mathtt{1}w)_7 \in S\}$*. The value on* $S \subseteq (\mathtt{10}\Sigma_7^*)_7$ *equals* $\varnothing$*.*

With Lemma 3 established and the expression (B.1) proved to implement the function $\mathit{Remove}_{\mathtt{1}}(X)$, the notation $\mathit{Remove}_{\mathtt{1}}(X)$ is used in equations to refer to this subexpression.

Next, consider the function $E(X)$ representing the existential quantifier ranging over strings in $\{\mathtt{3}, \mathtt{6}\}^*$. This function can be implemented by a single expression as follows:

Lemma E (Representing the existential quantifier)**.** *The value of the expression*

$$(X \cap (\mathtt{1}\Sigma_7^*)_7) \cup \big((X \cap (\{\mathtt{3}, \mathtt{6}\}^+\mathtt{1}\Sigma_7^*)_7)) + (-(\{\mathtt{3}, \mathtt{6}\}^+\mathtt{0}^*)_7) \cap (\mathtt{1}\Sigma_7^*)_7\big) \qquad (3.2)$$

on any $S \subseteq (\{\mathtt{3}, \mathtt{6}\}^*\mathtt{1}\Sigma_7^*)_7$ *is* $E(S) = \{(\mathtt{1}w)_7 \mid \exists w' \in \{\mathtt{3}, \mathtt{6}\}^*(w'\mathtt{1}w)_7 \in S\}$*.*

Note that $E(X)$ can already produce any recursively enumerable set from a recursive argument, and therefore it is essential to use subtraction in the expression.

With the existential quantifier implemented, the next task is to represent a universal quantifier. Ideally, one would be looking for an expression implementing $A(X)$, but, unfortunately, no such expression was found, and the actual construction given below implements the universal quantifier using multiple equations. The first step is devising an equation representing the function $f(X) = \{(x\mathtt{1}w)_7 \mid (\mathtt{1}w)_7 \in X\}$, which appends every string of digits in $\{\mathtt{3}, \mathtt{6}\}^*$ to numbers in its argument set.

Lemma 4. *For every constant set* $X \subseteq (\mathtt{1}\Sigma_7^*)_7$*, the equation*

$$Y = X \cup \mathit{Append}_{\mathtt{3},\mathtt{6}}(Y), \quad \textit{where}$$

$$\begin{aligned} \mathit{Append}_{\mathtt{3},\mathtt{6}}(Y) = &\bigcup_{i \in \{\mathtt{3},\mathtt{6}\}} \bigcup_{j \in \{\mathtt{3},\mathtt{6}\}} (Y \cap (j\Sigma_7^*)_7) + (\mathtt{20}^*)_7 \cap (\mathtt{2}j\Sigma_7^*)_7 + ((i-2)\mathtt{0}^*)_7 \cap (ij\Sigma_7^*)_7 \\ &\cup \bigcup_{i \in \{\mathtt{3},\mathtt{6}\}} (Y \cap (\mathtt{1}\Sigma_7^*)_7) + (i\mathtt{0}^*)_7 \cap (i\mathtt{1}\Sigma_7^*)_7 \end{aligned}$$

has the unique solution $Y = \{(x\mathtt{1}w)_7 \mid x \in \{\mathtt{3}, \mathtt{6}\}^*, (\mathtt{1}w)_7 \in X\}$*.*

Lemma A (Representing the universal quantifier)**.** *Let $S, \widetilde{S} \subseteq (\{\mathtt{3},\mathtt{6}\}^*\mathtt{1}\Sigma_7^*)_7$ be any sets, such that $\widetilde{S} \cap S = \varnothing$ and $(x\mathtt{1}w)_7 \in S$ and $(x'\mathtt{1}w)_7 \notin S$ implies $(x'\mathtt{1}w)_7 \in \widetilde{S}$. Then the following system of equations over sets of integers in variables Y, $\widetilde{Y}$ and Z*

$$Y = Z \cup \mathit{Append}_{\mathtt{3},\mathtt{6}}(Y) \tag{3.3}$$

$$\widetilde{Y} = E(\widetilde{S}) \cup \mathit{Append}_{\mathtt{3},\mathtt{6}}(\widetilde{Y}) \tag{3.4}$$

$$Z \subseteq (\mathtt{1}\Sigma_7^+)_7 \tag{3.5}$$

$$Y \subseteq S \subseteq Y \cup \widetilde{Y}, \tag{3.6}$$

has a unique solution

$$Z = A(S) = \{(\mathtt{1}w)_7 \mid \forall x:\ (x\mathtt{1}w)_7 \in S\}$$
$$Y = \{(y\mathtt{1}w)_7 \mid y \in \{\mathtt{3},\mathtt{6}\}^*,\ \forall x:\ (x\mathtt{1}w)_7 \in S\}$$
$$\widetilde{Y} = \{(y\mathtt{1}w)_7 \mid y \in \{\mathtt{3},\mathtt{6}\}^*,\ \exists x:\ (x\mathtt{1}w)_7 \in \widetilde{S}\}$$

Once the above quantifiers process a number $(\mathtt{1}x_k\mathtt{1}x_{k-1}\ldots\mathtt{1}x_1\mathtt{1}w)_7$, reducing it to $(\mathtt{1}w)_7$, it remains to obtain the actual number $(w)_7$ from this encoding.

Theorem 1. *Every arithmetical set $S \subseteq \mathbb{Z}$ ($S \subseteq \mathbb{N}$) is representable as a component of a unique solution of a system of equations over sets of integers (sets of natural numbers, respectively) with φ_j, ψ_j using the operations of addition and union and ultimately periodic constants (addition, substraction, union and singleton constants, respectively).*

4. Representing hyper-arithmetical sets

Following Moschovakis [14, Sec. 8E] and Aczel [1, Thm. 2.2.3], *hyper-arithmetical* sets $B_1, B_2, \ldots$ shall be defined as the *smallest effective σ-ring*, which is the recursive-theoretic counterpart to Borel sets (the smallest family of sets containing all open sets and closed under countable union and countable intersection).

Let $f_1, f_2, \ldots$ be an enumeration of all partial recursive functions and let τ_1, τ_2 be two recursive functions. Then, for all $k \in \mathbb{N}$,

$$B_{\tau_1(k)} = \mathbb{N} \setminus \{k\}, \quad C_{\tau_1(k)} = \{k\}$$

Moreover, for all numbers $k \in \mathbb{N}$, if f_k is a total function, then

$$B_{\tau_2(k)} = \bigcup_n C_{f_k(n)}, \quad C_{\tau_2(k)} = \bigcap_n B_{f_k(n)},$$

where the former operation is known as *effective σ-union*, while the latter is *effective σ-intersection*. Note that the only distinction between B_e and C_e is that the former is defined as a union and the latter as an intersection. As the definitions are dual, $B_e = \overline{C_e}$.

The family of sets $\mathcal{B} = \{B_e, C_e \mid e \in I \subseteq \mathbb{N}\}$ is called an effective σ-ring, if it contains $\{B_{\tau_1(e)}, C_{\tau_1(e)} \mid e \in \mathbb{N}\}$ and is closed under effective σ-union and effective σ-intersection. Then HA sets are defined as the smallest effective σ-ring. It is known [14, Sec. 8E] [1, Thm. 2.2.3] that for some (easy) choices of τ_1 and τ_2 the smallest effective σ-ring coincides with Δ_1^1 sets. We fix those two functions and the corresponding $\mathcal{B}$.

The smallest effective σ-ring can be defined as the least fixed point of a certain operator on the set $\mathcal{A} = 2^{\mathbb{N}\times 2^{\mathbb{N}}\times 2^{\mathbb{N}}}$, where a triple (e, B_e, C_e) indicates that the sets B_e and C_e have been defined for the index e in the above inductive definition, and an operator $\Phi : \mathcal{A} \to \mathcal{A}$

represents one step of this inductive definition. Furthermore, this least fixed point can be obtained constructively by a transfinite induction on countable ordinals, which is essential for any proofs about hyper-arithmetical sets.

With every set $B_e \in \mathcal{B}$ one can associate a *tree of* B_e, labeled with sets from $\mathcal{B}$: its root is labeled with B_e, and each vertex $B_{\tau_2(e')}$ ($C_{\tau_2(e')}$, respectively) in the tree has children labeled with $\{C_{f_{e'}(n)} \mid n \in \mathbb{N}\}$ ($\{B_{f_{e'}(n)} \mid n \in \mathbb{N}\}$, respectively). Vertices of the form $B_{\tau_1(e')}$ or $C_{\tau_1(e')}$ have no children and are thus leaves. The edge from the parent to the child is labeled by the child's label. A set X is said to be in the node if this node is labeled by X.

A partial order $\prec$ is *well-founded*, if it has no infinite descending chain. Extending this notion to oriented trees, a tree is well-founded if it contains no infinite downward path.

Lemma 5. *For each pair of sets $B_e, C_e \in \mathcal{B}$ the trees of B_e, C_e are well-founded.*

The well-foundedness of a set allows using the *well-founded induction principle*: given a property ϕ and a well founded order $\prec$ on set A

$$(\forall m \prec n\, \phi(m)) \Rightarrow \phi(n).$$

This principle shall be used in the below proof. Note, that the basis of the induction are $\prec$-minimal elements n of A, as for them $\phi(n)$ has to be shown directly.

Fix B_{i_0} as the target set in the root. Consider a path of length k in this tree, going from B_{i_0} to C_{i_1}, B_{i_2}, ..., B_{i_k} (or C_{i_k}, depending on the parity of k). Then, for each j-th set in this path, $i_{j-1} = \tau_2(n_j)$ and $i_j = f_k(n_j)$ for some number n_j, and the path is uniquely defined by the sequence of numbers $n_1, \ldots, n_k$. Consider the binary encoding of each of these numbers written using digits $\mathtt{3}$ and $\mathtt{6}$ (representing zero and one, respectively), and let *Resolve* be a partial function that maps finite sequences of such "binary" strings representing numbers $n_1, \ldots, n_k$ to the number i_k of the set B_{i_k} or C_{i_k} in the end of this path. The value of this function can be formally defined by induction:

$$\mathit{Resolve}(\langle\rangle) = i_0 \quad \mathit{Resolve}(x_1, \ldots, x_k) = f_{\tau_2^{-1}(\mathit{Resolve}(x_1, \ldots, x_{k-1}))}((x_k)_2),$$

Note that *Resolve* may be undefined if some τ_2-preimage is undefined.

The goal is to construct a system of equations, such that the following two sets are among the components of its unique solution:

$$\mathit{Goal}_0 = \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in B_{\mathit{Resolve}(x_1,\ldots,x_k)}\}$$
$$\mathit{Goal}_1 = \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in C_{\mathit{Resolve}(x_1,\ldots,x_k)}\}$$

These sets encode the sets $B_0, B_1, \ldots$ needed to compute B_{i_0}. In this way the (possibly infinite) amount of equations defining sets in hyper-arithmetical hierarchy is encoded in a finite amount of equations using only small number of variables. The set B_i in the node with path to the root encoded by $x_k, x_{k-1}, \ldots, x_1 \in \{\mathtt{3},\mathtt{6}\}^*$ is represented by $\{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_k\mathtt{10}w)_7 \mid (w)_7 \in B_i\} \subseteq \mathit{Goal}_0$.

The following set defines the admissible encodings, that is, numbers encoding paths in the tree of B_{i_0}:

$$\mathit{Admissible} = \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 | k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \mathit{Resolve}(x_1,\ldots,x_k) \text{ is defined}\}$$

The next two sets define the sets of leaves of tree of B_{i_0}, which are the constant sets B_i and C_i obtained by $i = \tau_1(e)$ for some e:

$$R_0 = \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0,\ x_i \in \{\mathtt{3},\mathtt{6}\}^*, \exists e \in \mathbb{N}, \mathit{Resolve}(x_1,\ldots,x_k) = \tau_1(e),\ (w)_7 \in B_{\tau_1(e)}\},$$
$$R_1 = \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0,\ x_i \in \{\mathtt{3},\mathtt{6}\}^*, \exists e \in \mathbb{N}, \mathit{Resolve}(x_1,\ldots,x_k) = \tau_1(e),\ (w)_7 \in C_{\tau_1(e)}\}$$

Lemma 6. *The sets $Goal_i$, $Admissible$, R_i are r.e. sets, $Resolve$ is an r.e. predicate.*

Consider the following system of equations.

$$X_0 = E(Remove_{\mathtt{1}}(X_1)) \cup R_0 \tag{4.1}$$
$$X_1 = Z \cup R_1 \tag{4.2}$$
$$\widetilde{Y} = E(Remove_{\mathtt{1}}(X_1)) \cup Append_{\mathtt{3},\mathtt{6}}(\widetilde{Y}) \tag{4.3}$$
$$Y = Z \cup Append_{\mathtt{3},\mathtt{6}}(Y) \tag{4.4}$$
$$Y \subseteq Remove_{\mathtt{1}}(X_0 \cap Admissible) \subseteq Y \cup \widetilde{Y} \tag{4.5}$$
$$Z \subseteq (\mathtt{1}\Sigma_7^+)_7 \tag{4.6}$$
$$X_0, X_1 \subseteq Admissible \tag{4.7}$$
$$X_0 \cap R_1 = \varnothing \tag{4.8}$$
$$X_1 \cap R_0 = \varnothing \tag{4.9}$$

Its intended unique solution will have $X_0 = Goal_0$ and $X_1 = Goal_1$, and will accordingly encode the set B_{i_0}, as well as all sets of $\mathcal{B}$ on which B_{i_0} logically depends. The system implements the functions $E(X)$ and $A(X)$ to represent effective σ-union and σ-intersection, respectively. For that purpose, the expression for $E(X)$ introduced in Lemma E, as well as the system of equations implementing $A(X)$ defined in Lemma A, are applied iteratively to the same variables X_0 and X_1. Simplifying the things, the above system may be regarded as an implementation of an equation $X_0 = A(E(X_0)) \cup const$.

The proof uses the principle of induction on well-founded structures. The membership of numbers of the form $(\mathtt{1}x_k\mathtt{1}x_{k-1}\ldots\mathtt{1}x_1\mathtt{10}w)_7$ in the variables X_0 and X_1, where $k \geqslant 0$, $x_i \in \{\mathtt{3},\mathtt{6}\}^*$ and $w \in \Sigma_7^* \setminus \mathtt{0}\Sigma_7^*$, is first proved for larger k's and then inductively extended down to $k = 0$, which allows extracting B_{i_0} out of the solution. The well-foundedness of the tree of B_{i_0} means that although B_{i_0} depends upon infinitely many sets, each dependency is over a finite path ending with a constant, that is, the self-dependence of numbers in X_0, X_1 on the numbers in X_0, X_1 reaches a constant R_0, R_1 in finitely many steps. This idea shall be formalised in the following lemmata.

Lemma 10 and Lemma 12 show that if there is a solution of system (4.1)–(4.9) then its solution has $X_0 = Goal_0$ and $X_1 = Goal_1$. What is left to prove is that those sets (together with some other sets) are in fact a solution.

Lemma 7. *The unique solution of the system* (4.1)–(4.9) *is*

$$X_0 = Goal_0 = \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in B_{Resolve(x_1,\ldots,x_k)}\}$$
$$X_1 = Goal_1 = \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in C_{Resolve(x_1,\ldots,x_k)}\}$$
$$Y = \{(x\mathtt{1}x_k\mathtt{1}x_{k-1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* (w)_7 \in B_{Resolve(x_1,\ldots,x_{k+1})}\}$$
$$\widetilde{Y} = \{(x\mathtt{1}x_k\mathtt{1}x_{k-1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* (w)_7 \in C_{Resolve(x_1,\ldots,x_{k+1})}\}$$
$$Z = Goal_1 \setminus R_1$$
$$= \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, e \in \mathbb{N}, x_i \in \{\mathtt{3},\mathtt{6}\}^*, Resolve(x_1,\ldots,x_k) = \tau_2(e), (w)_7 \in C_{\tau_2(e)}\}$$

The rest of the construction proceeds as follows: first, $Goal_0$ is intersected with $(\mathtt{10}\Sigma_7^*)_7$, which is a recursive set, and then the leading digits $\mathtt{10}$ are removed. The latter is done by analogy with the removal of $\mathtt{1}$ done in Lemma 3:

The expression $Remove_{10}$ works similarly to $Remove_1$, and the proof of this lemma is analogous to the proof of Lemma 3.

Theorem 2. *For every set $B \subseteq \mathbb{Z}$ in the hyper-arithmetical hierarchy there is a system of equations over subsets of $\mathbb{Z}$ using union, addition, singleton constants and the constants $\mathbb{N}$ and $-\mathbb{N}$, such that $(B, \ldots)$ is its unique solution.*

5. Equations with addition only

Equations over sets of natural numbers with addition as the only operation can represent an *encoding* of every recursive set, with each number $n \in \mathbb{N}$ represented by the number $16n + 13$ in the encoding. In order to define this encoding, for each $i \in \{0, 1, \ldots, 15\}$ and for every set $S \subseteq \mathbb{Z}$, denote:

$$\tau_i(S) = \{16n + i \mid n \in S\}.$$

The encoding of a set of natural numbers $S \subseteq \mathbb{N}$

$$S = \sigma_0(\widehat{S}) = \{0\} \cup \tau_6(\mathbb{N}) \cup \tau_8(\mathbb{N}) \cup \tau_9(\mathbb{N}) \cup \tau_{12}(\mathbb{N}) \cup \tau_{13}(\widehat{S}),$$

Proposition 4 ([9, Thm. 5.3])**.** *For every recursive set S there exists a system of equations over sets of natural numbers in variables $X, Y_1, \ldots, Y_m$ using the operation of addition and ultimately periodic constants, which has a unique solution with $X = \sigma_0(S)$.*

This result is proved by first representing the set S by a system with addition and union, and then by representing addition and union of sets using addition of their σ_0-encodings.

The purpose of this section is to obtain a similar result for equations over sets of integers: namely, that they can represent the same kind of encoding of every hyper-arithmetical set. For every set $\widehat{S} \subseteq \mathbb{Z}$, define its *encoding* as the set

$$S = \sigma(\widehat{S}) = \{0\} \cup \tau_6(\mathbb{Z}) \cup \tau_8(\mathbb{Z}) \cup \tau_9(\mathbb{Z}) \cup \tau_{12}(\mathbb{Z}) \cup \tau_{13}(\widehat{S}).$$

The subset $S \cap \{16n + i \mid n \in \mathbb{Z}\}$ is called the *ith track of S*.

The first result on this encoding is that the condition of a set X being an encoding of any set can be specified by an equation of the form $X + C = D$.

Lemma 8 (cf. [9, Lemma 3.3])**.** *A set $X \subseteq \mathbb{Z}$ satisfies an equation*

$$X + \{0, 4, 11\} = \bigcup_{\substack{i \in \{0,1,3,4,6,7,\\ 8,9,10,12,13\}}} \tau_i(\mathbb{Z}) \cup \{11\}$$

iff $X = \sigma(\widehat{X})$ for some $\widehat{X} \subseteq \mathbb{Z}$.

Now, assuming that the given system of equations with union and addition is decomposed to have all equations of the form $X = Y + Z$, $X = Y \cup Z$ or $X = const$, these equations can be simulated in a new system as follows:

Lemma 9 (cf. [9, Lemma 4.1])**.** *For all sets $X, Y, Z \subseteq \mathbb{Z}$,*

$$\sigma(Y) + \sigma(Z) + \{0, 1\} = \sigma(X) + \sigma(\{0\}) + \{0, 1\} \text{ iff } Y + Z = X$$

$$\sigma(Y) + \sigma(Z) + \{0, 2\} = \sigma(X) + \sigma(X) + \{0, 2\} \text{ iff } Y \cup Z = X.$$

Using these two lemmata, one can simulate any system with addition and union by a system with addition only. Taking systems representing different hyper-arithmetical sets, the following result on the expressive power of systems with addition can be established:

Theorem 3. *For every hyper-arithmetical set $S \subseteq \mathbb{Z}$ there exists a system of equations over sets of integers using the operation of addition and ultimately periodic constants, which has a unique solution with $X_1 = T$, where $S = \{n \mid 16n \in T\}$.*

6. Decision problems

Having a solution (solution existence) and having exactly one solution (solution uniqueness) are basic properties of a system of equations. For language equations with continuous operations, *solution existence* is Π^0_1-complete [19], and it remains Π^0_1-complete already in the case of a unary alphabet, concatenation as the only operation and regular constants [9], that is, for equations over sets of natural numbers with addition only. For the same formalisms, *solution uniqueness* is Π^0_2-complete.

Consider equations over sets of integers. Since their expressive power extends beyond the arithmetical hierarchy, the decision problems should accordingly be harder. In fact, the solution existence is Σ^1_1-complete, which will now be proved using a reduction from the following problem:

Proposition 5 (Rogers [21, Thm. 16-XX])**.** *Consider trees with nodes labeled by finite sequences of natural numbers, such that a node $(x_1, \ldots, x_{k-1}, x_k)$ is a son of $(x_1, \ldots, x_{k-1})$, and the empty sequence ε is the root. Then the following problem is Π^1_1-complete: "Given a description of a Turing machine accepting the set of nodes of a certain tree, determine whether this tree has no infinite paths".*

In other words, a given Turing machine recognizes sequences of natural numbers, and the task is to determine whether there is *no* infinite sequence of natural numbers, such that all of its prefixes would be accepted by the machine. The Σ^1_1-complete complement of the problem is testing whether such an infinite sequence exists, and it can be reformulated as follows:

Proposition 6. *The following problem is Σ^1_1-complete: "Given a Turing machine M working on natural numbers, determine whether there exists an infinite sequence of strings $\{x_i\}_{i=1}^{\infty}$ with $x_i \in \{\mathtt{3}, \mathtt{6}\}^*$, such that, for all $k \geqslant 0$, the number $(\mathtt{1}x_k\mathtt{1}x_{k-1}\ldots\mathtt{1}x_1\mathtt{1})_7$ is in $L(M)$".*

This problem will now be reduced to testing existence of a solution of equations over sets of numbers.

Theorem 4. *The problem of whether a given system of equations over sets of integers with addition and ultimately periodic constants has a solution is Σ^1_1-complete.*

Now consider the solution uniqueness property. The following upper bound on its complexity naturally follows by definition:

Proposition 7. *The problem of whether a given system of equations over sets of integers using addition and ultimately periodic constants has a unique solution can be represented as a conjunction of a Σ^1_1-formula and a Π^1_1-formula, and is accordingly in Δ^1_2.*

References

[1] P. Aczel, "An introduction to inductive definitions", in: J. Barwise (Ed.), *Handbook of Mathematical Logic*, 739–783, North-Holland, 1977.

[2] F. d'Alessandro, J. Sakarovitch, "The finite power property in free groups", *Theoretical Computer Science*, 293:1 (2003), 55–82.

[3] A. V. Anisimov, "Languages over free groups", *Mathematical Foundations of Computer Science*, (MFCS 1975, Mariánské Lázně, September 1–5, 1975), LNCS 32, 167–171.

[4] S. Ginsburg, H. G. Rice, "Two families of languages related to ALGOL", *Journal of the ACM*, 9 (1962), 350–371.

[5] J. Y. Halpern, "Presburger arithmetic with unary predicates is Π^1_1 complete", *Journal of Symbolic Logic*, 56:2 (1991), 637–642.

[6] A. Jeż, "Conjunctive grammars can generate non-regular unary languages", *International Journal of Foundations of Computer Science*, 19:3 (2008), 597–615.

[7] A. Jeż, A. Okhotin, "Conjunctive grammars over a unary alphabet: undecidability and unbounded growth", *Theory of Computing Systems*, to appear.

[8] A. Jeż, A. Okhotin, "On the computational completeness of equations over sets of natural numbers" *35th International Colloquium on Automata, Languages and Programming* (ICALP 2008, Reykjavik, Iceland, July 7–11, 2008), LNCS 5126, 63–74.

[9] A. Jeż, A. Okhotin, "Equations over sets of natural numbers with addition only", *STACS 2009* (Freiburg, Germany, 26–28 February, 2009), 577–588.

[10] M. Kunc, "The power of commuting with finite sets of words", *Theory of Computing Systems*, 40:4 (2007), 521–551.

[11] M. Kunc, "What do we know about language equations?", *Developments in Language Theory* (DLT 2007, Turku, Finland, July 3–6, 2007), LNCS 4588, 23–27.

[12] T. Lehtinen, A. Okhotin, "On equations over sets of numbers and their limitations", *Developments in Language Theory* (DLT 2009, Stuttgart, Germany, 30 June–3 July, 2009), LNCS 5583, 360–371.

[13] P. McKenzie, K. Wagner, "The complexity of membership problems for circuits over sets of natural numbers", *Computational Complexity*, 16:3 (2007), 211–244.

[14] Y. Moschovakis, *Elementary Induction on Abstract Structures*, North-Holland, 1974.

[15] A. Okhotin, "Conjunctive grammars", *Journal of Automata, Languages and Combinatorics*, 6:4 (2001), 519–535.

[16] A. Okhotin, "Conjunctive grammars and systems of language equations", *Programming and Computer Software*, 28:5 (2002), 243–249.

[17] A. Okhotin, "Unresolved systems of language equations: expressive power and decision problems", *Theoretical Computer Science*, 349:3 (2005), 283–308.

[18] A. Okhotin, "Computational universality in one-variable language equations", *Fundamenta Informaticae*, 74:4 (2006), 563–578.

[19] A. Okhotin, "Decision problems for language equations", *Journal of Computer and System Sciences*, to appear; earlier version at ICALP 2003.

[20] J. Robinson, "An introduction to hyperarithmetical functions", *Journal of Symbolic Logic*, 32:3 (1967), 325–342.

[21] H. Rogers, Jr., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, 1967.

[22] S. D. Travers, "The complexity of membership problems for circuits over sets of integers" *Theoretical Computer Science*, 369:1–3 (2006), 211–229.

Appendix.

Appendix A. Proofs from Section 2

Lemma 1.2. *Consider a system of equations over sets of integers, in variables $X_1, \ldots, X_n$, using the operations of union and addition, and any constant sets, which has a unique solution $X_i = S_i$. Then the same system, with each constant $C \subseteq \mathbb{Z}$ replaced by the set of opposite numbers $-C$, has the unique solution $X_i = -S_i$.* (p. 5)

Proof. Assume that the system is decomposed down to elementary equations $X_i + X_j = X_k$, $X_i \cup X_j = X_k$ and $X_i = C$. Make a negative version of the same system, with each constant C replaced by $-C$: it has variables $Y_1, \ldots, Y_n$.

It is sufficient to prove that if $(S_1, \ldots, S_n)$ is a solution of the former system, then $(-S_1, \ldots, -S_n)$ is a solution of the latter system (the converse claim is symmetric and holds by the same argument).

Consider an equation of the form $Y_i + Y_j = Y_k$ first. Then

$$S_i + S_j = \{s_i + s_j \mid s_i \in S_i, s_j \in S_j\} = S_k$$

In the new system

$$(-S_i)+(-S_j) = \{-s_i+(-s_j)| -s_i \in -S_i, -s_j \in -S_j\} = \{-(s_i+s_j)|s_i \in S_i, s_j \in S_j\} = -S_k$$

Similarly for other type of equations: consider equation of the form $Y_i \cup Y_j = Y_k$. Then

$$(-S_i) \cup (-S_j) = \{-s \mid -s \in -S_i \text{ or } -s \in -S_j\} = \{-s \mid s \in S_i \text{ or } s \in S_j\} = -S_k$$

For the equation of the form $Y_i = -C$ the argument is trivial. ∎

Lemma 1.3. *Let $R \subseteq \mathbb{N}$ be a recursive set. Then there exists a system of equations over sets of natural numbers using union, addition and singleton constants, which has variables $X, Y, Y', Z_1, \ldots, Z_m$, such that the set of solutions of this system is* (p. 5)

$$\left\{ (X = S,\ Y = S \cap R,\ Y' = S \cap \overline{R},\ Z_i = S_i) \;\middle|\; S \subseteq \mathbb{N} \right\},$$

where $S_1, \ldots, S_m$ are some fixed sets.

Proof. By Proposition 1, for each recursive set R (given by a TM T recognizing it and halting on every input) one can efficiently construct a system with a unique solution, such that R is one of its components. As the complement of a recursive set is effectively recursive, the set $\overline{R}$ is representable as well.

Add equations

$$Y \subseteq X \quad Y \subseteq R$$
$$Y' \cup Y = X \quad Y' \subseteq \overline{R}$$

Which are **clearly** equivalent to

$$Y = X \cap R \quad Y' = X \cap \overline{R}.$$

In particular there is a unique solution of the constructed system. ∎

Lemma 2. *If sets $S \cap \mathbb{N}$ and $(-S) \cap \mathbb{N}$ are representable by unique solutions of equations over sets of integers using union, addition, and ultimately periodic constants. Then S is representable by equations over integers using only union, addition and ultimately periodic constants.* (p. 5)

Proposition 8 ([16, Thm. 3])**.** *Let $X = \varphi(X)$ be an equation, where φ uses union, intersection, addition and constant sets of natural numbers, and furthermore, $\varphi(X)$ is of the form $\widehat{\varphi}(\varphi_1(X), \ldots, \varphi_k(X))$, where each $\varphi_i(X)$ is either a constant, or an expression of the form $\widetilde{\varphi}_i(X) + C$ with $C \subseteq \mathbb{N} \setminus \{0\}$. Then the equation has a unique solution.*

Appendix B. Proofs from Section 3

(p. 6) **Lemma 3.** *The value of the expression*

$$(X - \{1\} \cap \{0\}) \cup \bigcup_{i \in \Sigma_7 \setminus \{0\}} \bigcup_{t \in \{0,1\}} (X \cap (\mathtt{1}i\Sigma_7^t(\Sigma_7^2)^*)_7) - (\mathtt{10}^*)_7 \cap (i\Sigma_7^t(\Sigma_7^2)^*)_7 \tag{B.1}$$

on any $S \subseteq (\mathtt{1}(\Sigma_7^* \setminus \mathtt{0}\Sigma_7^*))_7$ is $Remove_{\mathtt{1}}(S) = \{(w)_7 \,|\, (\mathtt{1}w)_7 \in S\}$. The value on $S \subseteq (\mathtt{10}\Sigma_7^*)_7$ equals $\varnothing$.

Proof. Let ϕ denote the considered expression. By Proposition **??** it is distributive over infinite union, so we will eveluate it on a single number n. Suppose first that $n = 1$. Then clearly $\phi(\{1\}) = \{0\}$.

So suppose now that $n = (\mathtt{1}i'w)_7$. Observe, that if $i' = \mathtt{0}$ then the value of the expression is empty, as this number does not pass any intersection. Then let $i' \in \Sigma_7 \setminus \{\mathtt{0}\}$. The first term is empty, thus we focus only on the second one. The intersection with the set $(\mathtt{1}i\Sigma_7^t(\Sigma_7^2)^*)_7)$ is non-empty only for $i = i'$.

Consider first $t = 0$. Then n passes through the intersection with $(\mathtt{1}i\Sigma_7^t(\Sigma_7^2)^*)_7)$ only if it has odd number of digits (in base-7 notation). Then we subtract $m = (\mathtt{10}^\ell)_7$ from it and require that the result has an even amount of digit. As the second leading digit of n is non-zero, this can only happen if $\ell = |w| + 2$ and thus $n - m = (iw)_7$, for $|w|$ odd.

Consider now $t = 1$. By similar argument as for $t = 0$, we conclude that $\phi(\{(\mathtt{1}iw)_7\}) = \{(iw)_7\}$ for even $|w|$.

Hence

$$\phi((\{\mathtt{1}w)_7\}) = \{(w)_7\}$$

for $w \in \Sigma_7^* \setminus \mathtt{0}\Sigma_7^*$. ∎

(p. 6) **Lemma E.** *The value of the expression*

$$(X \cap (\mathtt{1}\Sigma_7^*)_7) \cup \big((X \cap (\{\mathtt{3},\mathtt{6}\}^+\mathtt{1}\Sigma_7^*)_7)) + (-(\{\mathtt{3},\mathtt{6}\}^+\mathtt{0}^*)_7) \cap (\mathtt{1}\Sigma_7^*)_7\big) \tag{B.2}$$

on any $S \subseteq (\{\mathtt{3},\mathtt{6}\}^*\mathtt{1}\Sigma_7^*)_7$ is $E(S) = \{(\mathtt{1}w)_7 \mid \exists w' \in \{\mathtt{3},\mathtt{6}\}^*(w'\mathtt{1}w)_7 \in S\}$.

Proof. Let ϕ denote the stated expression. Since it is, by Proposition **??**, distributive over infinite union, it is enough to consider the value of of ϕ on one number $n = (x\mathtt{1}y)_7 \in S$ for $x \in \{\mathtt{3},\mathtt{6}\}^*$. We consider the cases of $x = \varepsilon$ and $x \in \{\mathtt{3},\mathtt{6}\}^+$ separately.

If $n = (\mathtt{1}y)_7$, then it passes through intersection with $(\mathtt{1}\Sigma_7^*)_7$ in the first part of the union, and therefore it appears in the result. On the other hand, it is filtered out by the intersection in the second part of the union, hence

$$\phi(\{(\mathtt{1}y)_7\}) = \{(\mathtt{1}y)_7\} = E(\{(\mathtt{1}y)_7\})$$

Consider now $n = (x\mathtt{1}y)_7$ for $x \neq \varepsilon$. Then it is filtered by the first intersection, but it passes through the second, so it is enough to evaluate the second part of the union on this number.

Consider a number $(z\mathtt{0}^\ell)_7 \in (\{\mathtt{3},\mathtt{6}\}^*\mathtt{0}^*)_7$, where $z \in \{\mathtt{3},\mathtt{6}\}^*$. We show, that $(x\mathtt{1}y)_7 - (z\mathtt{0}^\ell)_7 \in (\mathtt{1}\Sigma_7^*)_7$ implies $x = z$ and $|\mathtt{1}y| = \ell$. This is done by comparing the position of the leftmost $\mathtt{1}$ in $x\mathtt{1}y$ and in the result.

We first show that a subtraction of $\mathtt{3}$ or $\mathtt{6}$ from $\mathtt{3}$ or $\mathtt{6}$ cannot create a digit $\mathtt{1}$, no matter what are the digits to the left and right. If we subtract $\mathtt{3}$ from $\mathtt{6}$ then the result is $\mathtt{3}$ or $\mathtt{2}$ ($\mathtt{2}$ can occure due to a possible borrowing). Similarly, when $\mathtt{3}$ is subtracted from $\mathtt{3}$, only $\mathtt{6}$ or $\mathtt{0}$ can occur. If $\mathtt{6}$ is subtracted from $\mathtt{6}$, only $\mathtt{0}$ or $\mathtt{6}$ can appear and finally, when $\mathtt{6}$ is subtracted from $\mathtt{3}$ only $\mathtt{4}$ or $\mathtt{3}$ can appear. So in ay case, $\mathtt{1}$ cannot be obtained.

In the same way, when $\mathtt{0}$ is subtracted from $\mathtt{3}$ or $\mathtt{6}$, $\mathtt{1}$ cannot be obtained, whatever are the digits to the left and right: if $\mathtt{0}$ is subtracted from $\mathtt{3}$, only $\mathtt{3}$ or $\mathtt{2}$ can be obtained. If from $\mathtt{6}$, only $\mathtt{6}$ or $\mathtt{5}$.

Consider the position p of the leftmost $\mathtt{1}$ in the result and in $x\mathtt{1}y$, $p' = |\mathtt{1}y|$. Then it cannot be that $p > p'$, as then $\mathtt{1}$ at position p' has to be obtained as a subtraction of $\mathtt{0}$ or $\mathtt{3}$ or $\mathtt{6}$ from $\mathtt{3}$ or $\mathtt{6}$. Suppose $p < p'$, so one of $\mathtt{3}$, $\mathtt{6}$ or $\mathtt{0}$ was subtracted from $\mathtt{1}$ and $\mathtt{0}$ was obtained. This can only appear when $\mathtt{0}$ is subtracted and there is a borrowing, but this is in fact not possible, as we subtract $(y\mathtt{0}^\ell)_7$, $y \in \{\mathtt{3},\mathtt{6}\}^*$ so if $\mathtt{0}$ is subtracted from $\mathtt{1}$, only $\mathtt{0}$ is subtracted from each position to the right, so in fact there is no need for a borrowing.

Thus $p = p'$. As $\mathtt{0}$ was subtracted from the leftmost $\mathtt{1}$ in $x\mathtt{1}y$, $(z\mathtt{0}^\ell)_7 \leqslant (x\mathtt{0}^\ell)_7$, otherwise the result is negative. If $(z\mathtt{0}^\ell)_7 < (x\mathtt{0}^\ell)_7$ then the leading digit is obtained on the position greater than p'. Thus $(z\mathtt{0}^\ell)_7 = (x\mathtt{0}^\ell)_7$ and therefore $(x\mathtt{1}y)_7 - (z\mathtt{0}^\ell)_7 = (\mathtt{1}y)_7$. Thus the result is in the desired set, i.e. a number $(\mathtt{1}y)_7$ is obtained, such that $(x\mathtt{1}y)_7 \in S$ for some $x \in \{\mathtt{3},\mathtt{6}\}^*$.

Hence for $x \in \{\mathtt{3},\mathtt{6}\}^+$

$$\phi(\{(x\mathtt{1}y)_7\}) = \{(\mathtt{1}y)_7\} = E(\{(x\mathtt{1}y)_7\}).$$

And so the assertion of the lemma holds. ∎

Lemma 4. *For every constant set* $X \subseteq (\mathtt{1}\Sigma_7^*)_7$*, the equation* (p. 6)

$$Y = X \cup Append_{\mathtt{3},\mathtt{6}}(Y),$$

where

$$Append_{\mathtt{3},\mathtt{6}}(Y) = \bigcup_{i\in\{\mathtt{3},\mathtt{6}\}} \bigcup_{j\in\{\mathtt{3},\mathtt{6}\}} (Y \cap (j\Sigma_7^*)_7) + (\mathtt{2}\mathtt{0}^*)_7 \cap (\mathtt{2}j\Sigma_7^*)_7 + ((i-2)\mathtt{0}^*)_7 \cap (ij\Sigma_7^*)_7$$
$$\cup \bigcup_{i\in\{\mathtt{3},\mathtt{6}\}} (Y \cap (\mathtt{1}\Sigma_7^*)_7) + (i\mathtt{0}^*)_7 \cap (i\mathtt{1}\Sigma_7^*)_7$$

has the unique solution $Y = \{(x\mathtt{1}w)_7 \mid x \in \{\mathtt{3},\mathtt{6}\}^*, (\mathtt{1}w)_7 \in X\}$.

Proof. First of all note that $Append_{\mathtt{3},\mathtt{6}}(Y) \subseteq \mathbb{N}$ for all Y: as $Append_{\mathtt{3},\mathtt{6}}(Y)$ intersects Y with a subset of natural numbers as its first operation and uses addition of natural numbers, intersections and union of subexpressions; all of those cannot introduce any negative numbers. Since $X \subseteq \mathbb{N}$, it follows that $Y \subseteq \mathbb{N}$ as well. Hence equation defining Y is in fact an equation over set of natural numbers.

The equation defining Y is a strict system, thus, by Proposition 8, it has a unique solution. So it is enough to show, that the specified Y is a solution. To this end we calculate

$X \cup \mathit{Append}_{3,6}(Y)$ and compare its value with Y. Since, by Proposition **??**, $\mathit{Append}_{3,6}$ is distributive, it is enough to calculate $\mathit{Append}_{3,6}(\{n\})$ for each $n \in Y$.

Consider the first big union. Fix i, j. Let $n = (j'w)_7$ for some $j' \in \Sigma$. Then $\{n\} \cap (j\Sigma_7^*)_7$ is non-empty only for $j' = j$. So in the following we assume that $j = j'$. Consider the term:

$$(\{n\} \cap (j\Sigma_7^*)_7) + (20^*)_7 \cap (2j\Sigma_7^*)_7.$$

We show that the result is $(2jw)_7$. Let $(20^m)_7$ be added to n. If $m > |w| + 1$ then $(jw)_7 + (20^m)_7 = (20^{m-|w|-1}jw)_7$. So the second leading digit is $0 \neq j$ and this number is filtered out by intersection with $(2j\Sigma_7^*)_7$. Suppose that $m \leq |w|$. Then the leading digit can be left intact, increased by 1 or a new leading digit 1 can be created. In all cases this is different from 2, so all those numbers are filtered out by intersection with $(2j\Sigma_7^*)_7$. If $m = |w| + 1$, the result is

$$(jw)_7 + (20^m)_7 = (2jw)_7,$$

as desired.

Consider now the second operation inside the first big term, i.e.

$$\{(2jw)_7\} + ((i-2)0^*)_7 \cap (ij\Sigma_7^*)_7$$

and suppose a number $((i-2)0^\ell)_7$ is added to $(2jw)_7$. If $\ell > |w| + 2$, the leading digit of the result is $(i-2)$ and is thus filtered out. If $\ell < |w| + 2$ then the leading digit is either 2 or 3. The former is filtered out. The latter is correct if $i = 3$. But in such case $(i-2)0^\ell = 10^\ell$ and so in order to have a carry $j = 6$ and is turned into 0, i.e. $(2jw)_7 + ((i-2)0^\ell)_7 = (26w)_7 + (10^\ell)_7 = (30w')_7$ and it is filtered out as well, since $j \neq 0$. Hence the only remaining subcase is $\ell = |w| + 2$:

$$(2jw)_7 + ((i-2)0^\ell)_7 = (ijw)_7.$$

And this is the wanted result. Taking the union over i, j we obtain, that $\mathit{Append}_{3,6}\{(jw)_7\} = (\{(3jw)_7, (6jw)_7\})_7$, whenever $j \in \{3, 6\}$.

Consider now $j = 1$. Then the first big term is empty. Let us consider the second one.

$$(\{n\} \cap (1\Sigma_7^*)_7) + ((i)0^*)_7 \cap (ij\Sigma_7^*)_7$$

Let a number $(i0^m)_7$ be added to $n = (1w)_7$. If $m > |w| + 1$ then $(i0^m)_7 + (1w)_7 = (i0^{m-|w|-1}1w)_7$, in particular the second last digit of it is 0, so it is filtered out by intersection with $(i1\Sigma_7^*)_7$. If $m < |w| + 1$ then the leading digit is 1 or 2, so it is also filtered out. If $m = |w| + 1$:

$$(i0^m)_7 + (1w)_7 = (i1w)_7,$$

as desired.

Hence $Append_{3,6}(\{(jw)_7\}) = (\{3jw, 6jw\})_7$, if $j \in \{1,3,6\}$ and $\varnothing$ otherwise. Using this in calculations of the right-hand side

$$\begin{aligned}
X \cup Append_{3,6}(Y) &= X \cup \bigcup_{n \in Y} Append_{3,6}(\{n\}) \\
&= X \cup \{Append_{3,6}(n) \mid n \in Y\} \\
&= X \cup \{(3w)_7, (6w)_7 \mid (w)_7 \in Y\} \\
&= X \cup \{(3w'1x)_7, (6w'1x)_7 \mid w' \in \{3,6\}^*,\ (1x)_7 \in X\} \\
&= X \cup \{(w'1x)_7 \mid w' \in \{3,6\}^+,\ (1x)_7 \in X\} \\
&= \{(w'1x)_7 \mid w' \in \{3,6\}^*,\ (1x)_7 \in X\} \\
&= Y
\end{aligned}$$

So Y is a unique solution of this equation. ■

Lemma A. *Let $S, \widetilde{S} \subseteq (\{3,6\}^* 1 \Sigma_7^*)_7$ be any sets, such that $\widetilde{S} \cap S = \varnothing$ and $(x1w)_7 \in S$ and $(x'1w)_7 \notin S$ implies $(x'1w)_7 \in \widetilde{S}$. Then the following system of equations over sets of integers in variables Y, $\widetilde{Y}$ and Z* (p. 7)

$$Y = Z \cup Append_{3,6}(Y) \tag{B.3}$$

$$\widetilde{Y} = E(\widetilde{S}) \cup Append_{3,6}(\widetilde{Y}) \tag{B.4}$$

$$Z \subseteq (1\Sigma_7^+)_7 \tag{B.5}$$

$$Y \subseteq S \subseteq Y \cup \widetilde{Y}, \tag{B.6}$$

has a unique solution

$$\begin{aligned}
Z &= A(S) = \{(1w)_7 \mid \forall x :\ (x1w)_7 \in S\} \\
Y &= \{(y1w)_7 \mid y \in \{3,6\}^*,\ \forall x :\ (x1w)_7 \in S\} \\
\widetilde{Y} &= \{(y1w)_7 \mid y \in \{3,6\}^*,\ \exists x :\ (x1w)_7 \in \widetilde{S}\}
\end{aligned}$$

Theorem 1. *Every arithmetical set $S \subseteq \mathbb{Z}$ ($S \subseteq \mathbb{N}$) is representable as a component of a unique solution of a system of equations* (p. 7)

$$\begin{cases}
\varphi_1(X_1, \dots, X_n) = \psi_1(X_1, \dots, X_n) \\
\vdots \\
\varphi_m(X_1, \dots, X_n) = \psi_m(X_1, \dots, X_n)
\end{cases}$$

over sets of integers (sets of natural numbers, respectively) with φ_j, ψ_j using the operations of addition and union and ultimately periodic constants (addition, substraction, union and singleton constants, respectively).

Since every arithmetical set is representable by a unique solution, Lemma 1.3 can now be strengthened to the following result to be used in the following:

Corollary 1 (Intersection with arithmetical constants)**.** *Let $R \subseteq \mathbb{N}$ be an arithmetical set. Then there is a system of equations over sets of natural numbers using union, addition and singleton constants, in variables $X, Y, Y', Z_1, \dots, Z_m$, such that the set of solutions of this system is*

$$\left\{ (X = S,\ Y = S \cap R,\ Y' = S \cap \overline{R},\ Z_i = S_i) \;\middle|\; S \subseteq \mathbb{N} \right\},$$

for some fixed sets $S_1, \dots, S_m$.

With this statement established, Lemma 1 can be accordingly improved to handle systems with arithmetical constants. Such systems shall now be used to represent an even greater family of sets.

Appendix C. Proofs from Section 4

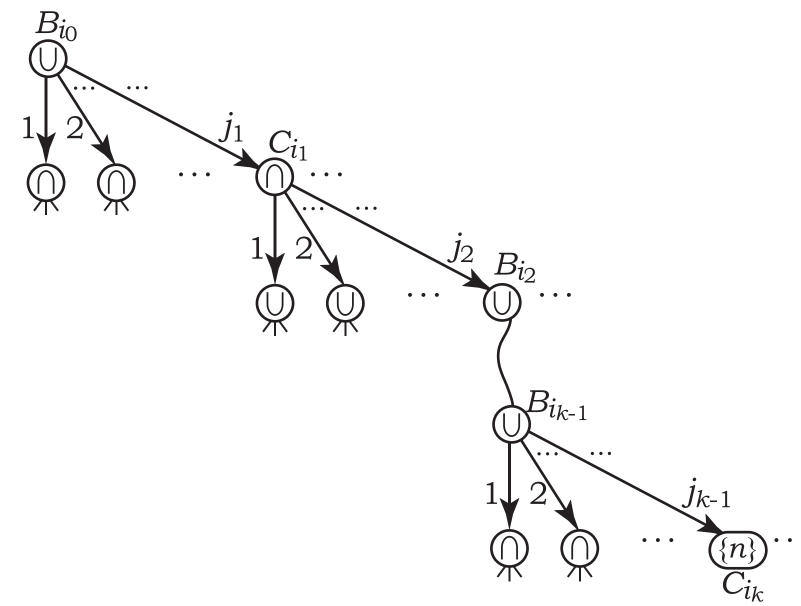

Figure 1: Definition of B_{i_0}.

(p. 8) **Lemma 5.** *For each pair of sets $B_e, C_e \in \mathcal{B}$ it holds that $B_e = \overline{C_e}$, and the trees of B_e, C_e are well-founded.*

Proof. First of all, note that, by the definition of B_e and C_e, the tree of B_e has the same structure as the tree of C_e, only with names of the vertices changed, i.e. with $B_{e'}$ replaced by $C_{e'}$ and $C_{e''}$ replaced by $B_{e''}$. So without loss of generality we may focus on trees for B_e.

In order to understand the proof, we have to define $\mathcal{B}$ in a more constructive manner. We follow the notation of [1], though it is essentially the same conventions as [14] stripped out of technical details.

Let $\mathcal{B}_0 = \{B_{\tau_1(e)}, C_{\tau_1(e)} \mid e \in \mathbb{N}\}$. For a successor ordinal $\lambda + 1$ define

$$\mathcal{B}_{\lambda+1} = \{B_{\tau_2(e)}, C_{\tau_2(e)} \mid \forall n\, C_{f_e(n)} \in \mathcal{B}_\lambda\} \cup \mathcal{B}_0$$

and for limit ordinal λ define

$$\mathcal{B}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{B}_\alpha$$

Then by standard set-theoretical tools and least-fixpoint theory it can be shown that $\mathcal{B}_{\omega_1} = \mathcal{B}_{\omega_1+1}$, where ω_1 is the least non-countable ordinal number. We take $\mathcal{B} = \mathcal{B}_{\omega_1}$. Then $\mathcal{B}_{\omega_1}$ is the least effective σ-ring [1, Ch. C7 Sec. 2.2] [14, Sec. 8E]. We show those results for completeness.

We show that if $\mathcal{B}_\alpha = \mathcal{B}_{\alpha+1}$ then $\mathcal{B}_\alpha$ is an effective σ-ring. Clearly it contains all the sets $\{B_{\tau_1(e)}, C_{\tau_1(e)} \mid e \in \mathbb{N}\}$. Moreover, if for e $\{C_{f_e(n)} \mid n \in \mathbb{N}\} \subseteq \mathcal{B}_\alpha$ then $B_{\tau_2(e)} \in \mathcal{B}_{\alpha+1} = \mathcal{B}_\alpha$.

Similarly $\{B_{f_e(n)} \,|\, n \in \mathbb{N}\} \subseteq \mathcal{B}_\alpha$ then $C_{\tau_2(e)} \in \mathcal{B}_{\alpha+1} = \mathcal{B}_\alpha$. Hence $\mathcal{B}_\alpha$ is closed udner effective σ-union and effective σ-intersection and it is an effective σ-ring. Moreover, if $\mathcal{B}_\alpha = \mathcal{B}_{\alpha+1}$ then $\mathcal{B}_\alpha = \mathcal{B}_\lambda$ for all $\lambda > \alpha$. This follows by standard transfinite induction argument: for limit ordinals $\lambda > \alpha$

$$\mathcal{B}_\lambda = \bigcup_{\lambda' < \lambda} \mathcal{B}_{\lambda'} \subseteq \bigcup_{\lambda' < \lambda} \mathcal{B}_\alpha = \mathcal{B}_\alpha$$

and for successor ordinals $\lambda = \lambda' + 1 > \alpha$:

$$\begin{aligned} \mathcal{B}_\lambda &= \{B_{\tau_2(e)}, C_{\tau_2(e)} \mid \forall n\, C_{f_e(n)}, B_{f_e(n)} \in \mathcal{B}_{\lambda'}\} \cup \mathcal{B}_0 \\ &= \{B_{\tau_2(e)}, C_{\tau_2(e)} \mid \forall n\, C_{f_e(n)}, B_{f_e(n)} \in \mathcal{B}_{\alpha}\} \cup \mathcal{B}_0 \\ &\subseteq \mathcal{B}_{\alpha+1} \\ &= \mathcal{B}_\alpha \end{aligned}$$

We show that for some $\lambda < \omega_1$ it holds that $\mathcal{B}_\lambda = \mathcal{B}_{\lambda+1}$. Suppose not. Then for all $\lambda < \omega_1$ $\mathcal{B}_{\lambda+1} \setminus \mathcal{B}_\lambda$ is non-empty. Define $D^\lambda \in \mathcal{B}_{\lambda+1} \setminus \mathcal{B}_\lambda$. Then $|\mathcal{B}_{\omega_1}| \geq |\{D^\lambda \mid \lambda < \omega_1\}| = \aleph_1 > \aleph_0$ and this is a contradiction, as $\mathcal{B}_{\omega_1} \subseteq \{B_e, C_e \mid e \in \mathbb{N}\}$, i.e. it has countably many elements.

We use the constructive definition of $\mathcal{B}$ to show that tree of B_e is always well-founded: consider, for the sake of contradiction, $\mathcal{B}_\lambda$ such that there exists $B_e \in \mathcal{B}_\lambda$ such that the tree of B_e is not well founded. Since the ordinals are well-founded, there exists a minimal such λ.

Ordinal λ cannot be a limit ordinal, as then

$$\mathcal{B}_\lambda = \bigcup_{\alpha < \lambda} \mathcal{B}_\alpha$$

and therefore $B_e \in \mathcal{B}_\alpha$ for $\alpha < \lambda$, contradiction.

Similarly $\lambda \neq 0$, as all the sets from $\mathcal{B}_0$ have trees of 0-height.

So $\lambda = \alpha + 1$. Let the children of the root of the tree of B_e be $C_{e_1}, \ldots, C_{e_n}, \ldots$. Then one of them, say C_{e_i} is not well-founded. But it means that B_{e_i} has a tree which is not well-founded either, and $B_{e_i} \in \mathcal{B}_\alpha$, contradiction.

Hence all the trees in $\mathcal{B}$ are well-founded.

Similar argument by transfinite induction is applied to show that $B_e = \overline{C_e}$. Consider λ such that there are $B_e, C_e \in \mathcal{B}_\lambda$ such that $B_e \neq \overline{C_e}$. As ordinals are well-founded, we can choose a minimal such λ. Then clearly $\lambda \neq 0$, as by definition $B_{\tau_1(k)} = \{k\} = \overline{C_{\tau_1(k)}}$.

Suppose that λ is a limit ordinal. But then $B_e, C_e \in \mathcal{B}_\alpha$ for some $\alpha < \lambda$.

So assume that $\lambda = \alpha + 1$. Then

$$B_e = B_{\tau_2(k)} = \bigcup_n C_{f_k(n)} \quad C_e = C_{\tau_2(k)} = \bigcap_n B_{f_k(n)}$$

and be definition, for all n $C_{f_k(n)}, B_{f_k(n)} \in \mathcal{B}_\alpha$. Thus $C_{f_k(n)} = \overline{B_{f_k(n)}}$ and hence

$$C_{\tau_2(k)} = \bigcap_n B_{f_k(n)} = \bigcap_n \overline{C_{f_k(n)}} = \overline{\bigcup_n C_{f_k(n)}} = \overline{B_{\tau_2(k)}}.$$

Which concludes this part of the proof. ∎

Lemma 6. *The sets $Goal_i$, Admissible, R_i are r.e. sets, Resolve is an r.e. predicate.* (p. 9)

Proof. Consider first that $Resolve(x_1, \dots, x_k)$ can be represented in the form $(\exists i_1) \dots (\exists i_k)(\exists \ell_1) \dots (\exists \ell_k)(\exists t)R$, where R is a recursive predicate that determines whether for every $j \in \{1, \dots, k\}$, $\tau_2(\ell_j) = i_{j-1}$ and $i_j = f_{\ell_j}((x_j)_2)$, and furthermore, that the evaluation of each f_{ℓ_j} is done in at most t steps of computation. This proves that *Resolve* is recursively enumerable.

We give the proof for R_0, the other sets are similar. To determine, whether $(\mathtt{1}x_k\mathtt{1}x_{k-1}\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \in R_0$ we first calculate the value $i_k = Resolve(x_1, \dots, x_k)$. This can be done, as *Resolve* is a RE predicate. Then we check whether $\tau_1^{-1}(i_k) = (w)_7$. ∎

Lemma 10. *If $Resolve(x_1, \dots, x_k) = \tau_1(e)$ then*

$$\begin{aligned} X_0 \cap (\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 &= \{(\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in B_{\tau_1(e)}\} \\ X_1 \cap (\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 &= \{(\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in C_{\tau_1(e)}\} \end{aligned}$$

Proof. By (4.1) $R_0 \subseteq X_0$

$$X_0 \cap (\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 \supseteq R_0 \cap (\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = \{(\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in B_{\tau_1(e)}\}$$

On the other hand, by (4.8)

$$\begin{aligned} X_0 \cap \{(\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \notin B_{\tau_1(e)}\} &= X_0 \cap \{(\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in C_{\tau_1(e)}\} \\ &= X_0 \cap R_1 \\ &= \varnothing \end{aligned}$$

Hence the claim for X_0 follows. Similar calculations can be done for X_1: by (4.2) $R_1 \subseteq X_1$, therefore

$$X_1 \cap (\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 \supseteq R_1 \cap (\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = \{(\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in C_{\tau_1(e)}\}$$

Conversely, by (4.9)

$$\begin{aligned} X_1 \cap \{(\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \notin C_{\tau_1(e)}\} &= X_1 \cap \{(\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in B_{\tau_1(e)}\} \\ &= X_1 \cap R_0 \\ &= \varnothing \end{aligned}$$

And hence

$$X_1 \cap (\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = \{(\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in C_{\tau_1(e)}\}$$

∎

Lemma 11. *If $Resolve(x_1, \dots, x_k)$ is defined, then every solution $(X_0, X_1, \dots ...)$ of* (4.1)–(4.9) *satisfies*

$$X_1 \cap (\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = \{(\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \mid (\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \notin X_0\}$$

Moreover if there is e such that $Resolve(w_1, \dots, w_k) = \tau_2(e)$, then

$$\begin{aligned} X_0 \cap (\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 &= \{(\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \mid \exists x_{k+1} \in \{\mathtt{3}, \mathtt{6}\}^* \ (\mathtt{1}x_{k+1}\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \in X_1\} \\ X_1 \cap (\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 &= \{(\mathtt{1}x_k\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \mid \forall x_{k+1} \in \{\mathtt{3}, \mathtt{6}\}^* \ (\mathtt{1}w_{x+1}\mathtt{1} \dots \mathtt{1}x_1\mathtt{10}w)_7 \in X_0\} \end{aligned}$$

Proof. In order to show the claim we proceed by induction. We exploit the inductive definition of HA sets: consider an ordering $\prec$ of sets $\{X \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma^*)_7 \mid \mathit{Resolve}(x_1,\ldots,x_k) \text{ is defined}\}$ defined by saying that $X \cap (\mathtt{1}x_{k+1}\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma^*)_7$ is a direct predecessor of $X \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma^*)_7$. Note that each descending sequence of sets corresponds to a path in the tree of B_{i_0}. Since this tree is well founded by Lemma 5, also $\prec$ is well-founded. Thus we may prove the theorem using an induction principle, i.e. show that if the thesis holds for all direct predecessors of $X \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma^*)_7$, then it holds for $X \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma^*)_7$ as well. Note, that some sets have no predecessors — being more precise, those that $\mathit{Resolve}(x_1,\ldots,x_k) = \tau_1(e)$ for some e.

Consider first the induction basis, i.e. $(x_1,\ldots,x_k)$ such that $\mathit{Resolve}(x_1,\ldots,x_k)$ is defined but for any x_{k+1} $\mathit{Resolve}(x_1,\ldots,x_{k+1})$ is not. By definition this means that $\mathit{Resolve}(x_1,\ldots,x_k) = \tau_1(e)$ for some e. Then by Lemma 10

$$\begin{aligned} X_0 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 &= \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in B_{\tau_1}(e)\} \\ X_1 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 &= \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in C_{\tau_1}(e)\} \end{aligned}$$

Hence

$$\begin{aligned} X_1 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 &= \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in C_{\tau_1}(e)\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \notin B_{\tau_1}(e)\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \notin X_0\} \end{aligned}$$

We now prove the induction step — we fix $w_1,\ldots,w_k$, such that $\mathit{Resolve}(w_1,\ldots,w_k) = \tau_2(e)$, and assume that all the claims of the lemma hold for all $X_0 \cap (\mathtt{1}x_{k+1}\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7$ and $X_1 \cap (\mathtt{1}x_{k+1}\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7$. Then we show that they hold for $X_0 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7$ and $X_1 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7$ as well.

Our goal is to extract the equations defining the two sets in question out of the system and infer the needed properties. This is done by intersecting the system with specially chosen constants and applying Lemma E and Lemma A to the newly obtained system.

Intersect (4.1) with set $(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7$. Note that

$$R_0 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = \varnothing$$

as $\mathit{Resolve}(x_1,\ldots,x_k) = \tau_2(e)$.

Then consider

$$E(\mathit{Remove}_{\mathtt{1}}(X_1)) \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7$$

Since

$$E(\mathit{Remove}_{\mathtt{1}}(X_1)) = \{(\mathtt{1}w)_7 \mid \exists x \in \{\mathtt{3},\mathtt{6}\}^* \ (\mathtt{1}x\mathtt{1}w)_7 \in X_1\}$$

we obtain that

$$\begin{aligned} &E(\mathit{Remove}_{\mathtt{1}}(X_1)) \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 \\ &= \{(\mathtt{1}w)_7 \mid \exists x \in \{\mathtt{3},\mathtt{6}\}^* \ (\mathtt{1}x\mathtt{1}w)_7 \in X_1\} \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 \\ &= \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* \ (\mathtt{1}x_{k+1}\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \in X_1\} \end{aligned}$$

but by induction assumption

$$\begin{aligned}(\mathtt{1}x_{k+1}\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \in X_1 &\iff (w)_7 \in C_{Resolve(x_1,\dots,x_{k+1})} \\ &\iff (w)_7 \in C_{f_{\tau_2^{-1}(Resolve(x_1,\dots,x_k))}((x_{k+1})_2)} \\ &\iff (w)_7 \in C_{f_{\tau_2^{-1}(\tau_2(e))}((x_{k+1})_2)} \\ &\iff (w)_7 \in C_{f_{e((x_{k+1})_2)}}\end{aligned}$$

Thus

$$\begin{aligned}&\{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* \ (\mathtt{1}x_{k+1}\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \in X_1\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* \ (w)_7 \in C_{f_{e((x_{k+1})_2)}}\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in \bigcup_{x_{k+1}\in\{\mathtt{3},\mathtt{6}\}^*} C_{f_{e((x_{k+1})_2)}}\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in B_{\tau_2(e)}\}\end{aligned}$$

And so the claim follows.

We now use the same approach in a little more complicated case: we want to "apply" Lemma A to (4.3)–(4.6). As we are interested now only in (more or less) subsets of $(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7$, we intersect the equations with appropriate sets. Then we obtain other equations, to which Lemma A is applicable — the needed technical assumptions of the lemma will be satisfied by the induction assumption.

Introduce new variables X_0', X_1', Y', $\widetilde{Y}'$, Z'. They are related to the existing variables by equations

$$\begin{aligned}X_0' &= X_0 \cap (\mathtt{1}\{\mathtt{3},\mathtt{6}\}^*\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 \\ X_1' &= X_1 \cap (\mathtt{1}\{\mathtt{3},\mathtt{6}\}^*\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 \\ Y' &= Y \cap (\{\mathtt{3},\mathtt{6}\}^*\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 \\ \widetilde{Y}' &= \widetilde{Y} \cap (\{\mathtt{3},\mathtt{6}\}^*\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 \\ Z' &= Z \cap (\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7\end{aligned}$$

Then, for a fixed solution of the system, those variables are assigned fixed values. We infer a new system of equations, using only and the new variables and constants. Then we apply Lemma E and Lemma A to the new system.

Consider the intersection of (4.3) with set $(\{\mathtt{3},\mathtt{6}\}^*\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7$. Then on the left-hand side we obtain $\widetilde{Y} \cap (\{\mathtt{3},\mathtt{6}\}^*\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = \widetilde{Y}'$ while on the right-hand side

$$\Big(E(Remove_{\mathtt{1}}(X_1)) \cup Append_{\mathtt{3},\mathtt{6}}(\widetilde{Y})\Big) \cap (\{\mathtt{3},\mathtt{6}\}^*\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7$$

Let us calculate them separately:

$$\begin{aligned}E(Remove_{\mathtt{1}}(X_1)) \cap (\{\mathtt{3},\mathtt{6}\}^*\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 &= E(Remove_{\mathtt{1}}(X_1)) \cap (\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 \\ &= E(Remove_{\mathtt{1}}(X_1) \cap (\{\mathtt{3},\mathtt{6}\}^*\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7) \\ &= E(Remove_{\mathtt{1}}(X_1 \cap (\mathtt{1}\{\mathtt{3},\mathtt{6}\}^*\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7)) \\ &= E(Remove_{\mathtt{1}}(X_1'))\end{aligned}$$

and

$$\begin{aligned}Append_{3,6}(\widetilde{Y}) \cap (\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7 &= Append_{3,6}(\widetilde{Y}) \cap (\{3,6\}^+ 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7 \\ &= Append_{3,6}(\widetilde{Y} \cap (\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7) \\ &= Append_{3,6}(\widetilde{Y}')\end{aligned}$$

And therefore the following equation is obtained

$$\widetilde{Y}' = E(Remove_1(X_1')) \cup Append_{3,6}(\widetilde{Y}')$$

Now intersect (4.4) with the same set, the calculations are similar to the ones for (4.3):

$$\begin{aligned}&Y \cap (\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7 \\ &= Y' \\ &= (Z \cap (\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7) \cup (Append_{3,6}(Y) \cap (\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7) \\ &= (Z \cap (\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7) \cup (Append_{3,6}(Y) \cap (\{3,6\}^+ 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7) \\ &= (Z \cap (1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7) \cup Append_{3,6}(Y \cap (\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7) \\ &= Z' \cup Append_{3,6}(Y')\end{aligned}$$

And lastly consider an intersection of (4.5) with the same set

$$\begin{aligned}Y \cap (\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7 &\subseteq Remove_1(X_0 \cap Admissible) \cap (\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7 \\ &= Remove_1(X_0 \cap (1\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7 \cap Admissible) \\ &\subseteq Remove_1(X_0 \cap (1\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7) \\ &\subseteq Y \cap (\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7 \cup \widetilde{Y} \cap (\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7\end{aligned}$$

Thus we obtain

$$Y' \subseteq Remove_1(X_0') \subseteq Y' \cup \widetilde{Y}'$$

The newly obtained equations, together with

$$Z' \subseteq Z \subseteq (1\Sigma_7^+)_7$$

inferred from (4.6), satisfy the assumption of the Lemma A — it is enough for us to check the technical conditions imposed on X_0' and X_1'. First of all we check that they are disjoint:

$$\begin{aligned}X_0' \cap X_1' &= X_0 \cap X_1 \cap (1\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7) \\ &= \bigcup_{x_{k+1} \in \{3,6\}^*} X_0 \cap X_1 \cap (1 x_{k+1} 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7) \\ &= \bigcup_{x_{k+1} \in \{3,6\}^*} \varnothing\end{aligned}$$

with the last equation following from the induction assumption.

Then consider $x_{k+1}, x_{k+1}' \in \{3,6\}^*$ and w such that $(1 x_{k+1} 1 w)_7 \in X_0 \cap (1\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7$ and $(1 x_{k+1}' 1 w)_7 \notin X_0 \cap (1\{3,6\}^* 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7$. Then $w = 1 x_k 1 \dots 1 x_1 1 0 w'$ and therefore $1 x_{k+1} 1 x_k 1 \dots 1 x_1 1 0 w' \in X_0$ and $1 x_{k+1}' 1 x_k 1 \dots 1 x_1 1 0 w' \notin X_0$. Then by induction assumption

$$X_1' = X_1 \cap (1 x_{k+1} 1 x_k 1 \dots 1 x_1 1 0 \Sigma_7^*)_7 = \{(1 x_{k+1} 1 x_k 1 \dots 1 x_1 1 0 w'')_7 | (1 x_{k+1} 1 x_k 1 \dots 1 x_1 1 0 w'')_7 \notin X_0\}$$

in particular $(1 x_{k+1}' 1 x_k 1 \dots 1 x_1 1 0 w')_7 \in X_1$, as required by Lemma A.

Since the assumption of Lemma A are met,

$$Z' = Z \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = \{(\mathtt{1}w)_7 \mid \forall x(\mathtt{1}x\mathtt{1}w)_7 \in X_0 \cap (\mathtt{1}\{\mathtt{3},\mathtt{6}\}^*\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7\}$$
$$= \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid \forall x_{k+1}(\mathtt{1}x_{k+1}\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \in X_0\}$$

Then intersecting (4.2) with set $(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7$ yields:

$$X_1 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = (Z \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7) \cup (R_1 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7)$$
$$= \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid \forall x_{k+1}(\mathtt{1}x_{k+1}\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \in X_0\} \cup \varnothing$$

and so the third claim of the lemma follows.

We need just to check the first claim of the lemma:

$$X_1 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \notin X_0\}$$

Let us calculate:

$$X_1 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*\ (\mathtt{1}w_{x+1}\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \in X_0\}$$
$$= \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*\ (\mathtt{1}w_{x+1}\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \notin X_1\}$$
$$= \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid \neg(\exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*\ (\mathtt{1}w_{x+1}\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \in X_1)\}$$
$$= \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid \neg((\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \in X_0)\}$$
$$= \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \notin X_0\}$$

∎

Lemma 12. *If* $Resolve(x_1,\ldots,x_k) = e$ *is defined, then*

$$X_0 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in B_e\} \quad \text{(C.1)}$$
$$X_1 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in C_e\} \quad \text{(C.2)}$$

Proof. As in the previous lemma, we employ the $\prec$ ordering of sets and apply the notion of induction on well-founded structures.

For base sets, i.e. the ones without a successor, it holds that $Resolve(x_1,\ldots,x_k) = \tau_1(e)$. Then by Lemma 10

$$X_0 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in B_{\tau_1}(e)\}$$
$$X_1 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in C_{\tau_1}(e)\}$$

i.e. the claim holds.

So consider now $x_1,\ldots,x_k$ such that $Resolve(x_1,\ldots,x_k) = \tau_2(e)$ for some e. Then by Lemma 11

$$X_0 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*\ (\mathtt{1}x_{k+1}\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \in X_1\}$$
$$X_1 \cap (\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 = \{(\mathtt{1}x_k\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \mid \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*\ (\mathtt{1}x_{k+1}\mathtt{1}\ldots\mathtt{1}x_1\mathtt{10}w)_7 \in X_0\}$$

and by induction assumption this is equal to

$$\begin{aligned}X_0 \cap (\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* \ (\mathtt{1}x_{k+1}\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \in X_1\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* \ (w)_7 \in C_{Resolve(x_1,\dots,x_{k+1})}\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \bigcup_{x_{k+1}\in\mathbb{N}} \in C_{f_{\tau_2^{-1}(Resolve(x_1,\dots,x_k))}(x_{k+1})}\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \bigcup_{x_{k+1}\in\mathbb{N}} \in C_{f_e(x_{k+1})}\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in B_{\tau_2(e)}\}\end{aligned}$$

Similar calculations can be done for X_1:

$$\begin{aligned}X_1 \cap (\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* \ (\mathtt{1}x_{k+1}\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \in X_0\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* \ (w)_7 \in B_{Resolve(x_1,\dots,x_{k+1})}\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in \bigcap_{x_{k+1}\in\mathbb{N}} B_{f_{\tau_2^{-1}(Resolve(x_1,\dots,x_{k+1}))}(x_{k+1})}\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in \bigcap_{x_{k+1}\in\mathbb{N}} B_{f_e(x_{k+1})}\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid (w)_7 \in C_{\tau_2(e)}\}\end{aligned}$$

■

Lemma 7. *The unique solution of the system* (4.1)–(4.9) *is* (p. 9)

$$\begin{aligned}X_0 &= Goal_0 = \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in B_{Resolve(x_1,\dots,x_k)}\} \\ X_1 &= Goal_1 = \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in C_{Resolve(x_1,\dots,x_k)}\} \\ Y &= \{(x\mathtt{1}x_k\mathtt{1}x_{k-1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* (w)_7 \in B_{Resolve(x_1,\dots,x_{k+1})}\} \\ \widetilde{Y} &= \{(x\mathtt{1}x_k\mathtt{1}x_{k-1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* (w)_7 \in C_{Resolve(x_1,\dots,x_{k+1})}\} \\ Z &= Goal_1 \setminus R_1 \\ &= \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, e \in \mathbb{N}, x_i \in \{\mathtt{3},\mathtt{6}\}^*, Resolve(x_1,\dots,x_k) = \tau_2(e), (w)_7 \in C_{\tau_2(e)}\}\end{aligned}$$

Proof. We give a routinous calculations

Consider first (4.1). By Lemma E

$$\begin{aligned}&E(Remove_1(Goal_1)) \\ &= E(\{(x_k\mathtt{1}x_{k-1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in C_{Resolve(x_1,\dots,x_k)}\}) \\ &= \{(\mathtt{1}x_{k-1}\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \exists x_k \in \{\mathtt{3},\mathtt{6}\}^* \ (w)_7 \in C_{Resolve(x_1,\dots,x_k)}\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* \ (w)_7 \in C_{Resolve(x_1,\dots,x_k,x_{k+1})}\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in \bigcup_{x_{k+1}\in\{\mathtt{3},\mathtt{6}\}^*} C_{f_{\tau_2^{-1}(Resolve(x_1,\dots,x_k))}}((x_{k+1})_2)\} \\ &= \{(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, \ x_i \in \{\mathtt{3},\mathtt{6}\}^*, \ \exists e \in \mathbb{N}, \ Resolve(x_1,\dots,x_k) = \tau_2(e), \ (w)_7 \in B_{\tau_2(e)}\}\end{aligned}$$

And as

$$R_0 = \{(\mathtt{1}x_k\dots\mathtt{1}x_1\mathtt{10}w)_7 | k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \exists e \in \mathbb{N}, Resolve(x_1,\dots,x_k) = \tau_1(e), (w)_7 \in B_{\tau_1(e)}\}$$

We obtain that

$$\begin{aligned}E(Remove_1(Goal_1)) \cup R_0 =& \{(\mathtt{1}x_k \dots \mathtt{1}x_1\mathtt{1}\mathtt{0}w)_7 \mid k \geqslant 1,\ x_i \in \{\mathtt{3},\mathtt{6}\}^*,\ \exists e \in \mathbb{N}\\ &Resolve(x_1,\dots,x_k) = \tau_1(e),\ (w)_7 \in B_{\tau_1(e)} \text{ or}\\ &Resolve(x_1,\dots,x_k) = \tau_2(e),\ (w)_7 \in B_{\tau_2(e)}\}\\ =&\{(\mathtt{1}x_k \dots \mathtt{1}x_1\mathtt{1}\mathtt{0}w)_7 \mid k \geqslant 1, x_i \in \{\mathtt{3},\mathtt{6}\}^*,\ (w)_7 \in B_{Resolve(x_1,\dots,x_k)}\}\\ =&Goal_0\end{aligned}$$

So we have shown that (4.1) holds.

Consider now (4.2). As $R_1 \subseteq Goal_1$

$$(Goal_1 \setminus R_1) \cup R_1 = Goal_1$$

and so (4.2) holds.

Consider (4.3). Then by Lemma E, similarly to the calculations done for (4.1),

$$E(Remove_1(Goal_1)) =$$
$$\{(\mathtt{1}x_k\mathtt{1}x_{k-1}\dots \mathtt{1}x_1\mathtt{1}\mathtt{0}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* (w)_7 \in C_{Resolve(x_1,\dots,x_{k+1})}\}.$$

Considering $Append_{\mathtt{3},\mathtt{6}}(\widetilde{Y})$, by Lemma 4

$$Append_{\mathtt{3},\mathtt{6}}(\widetilde{Y}) =$$
$$\{(x\mathtt{1}x_k\mathtt{1}x_{k-1}\dots \mathtt{1}x_1\mathtt{1}\mathtt{0}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, x \in \{\mathtt{3},\mathtt{6}\}^+\ \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*\ (w)_7 \in C_{Resolve(x_1,\dots,x_k,x_{k+1})}\}$$

Therefore the right hand side of (4.3) is equal to

$$\begin{aligned}&\{(\mathtt{1}x_k\mathtt{1}x_{k-1}\dots \mathtt{1}x_1\mathtt{1}\mathtt{0}w)_7 \mid k \geqslant 0,\ x_i \in \{\mathtt{3},\mathtt{6}\}^*,\ \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*\ (w)_7 \in C_{Resolve(x_1,\dots,x_{k+1})}\}\\ &\cup \{(x\mathtt{1}x_k\mathtt{1}x_{k-1}\dots \mathtt{1}x_1\mathtt{1}\mathtt{0}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, x \in \{\mathtt{3},\mathtt{6}\}^+,\ \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*\ (w)_7 \in C_{Resolve(x_1,\dots,x_{k+1})}\}\\ &= \{(x\mathtt{1}x_k\mathtt{1}x_{k-1}\dots \mathtt{1}x_1\mathtt{1}\mathtt{0}w)_7 \mid k \geqslant 0, x, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*\ (w)_7 \in C_{Resolve(x_1,\dots,x_{k+1})}\}\end{aligned}$$

and thus (4.3) follows.

Consider (4.4) we calculate the two parts separately:

$$\begin{aligned}&(Goal_1 \setminus R_1)\\ &= \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\dots \mathtt{1}x_1\mathtt{1}\mathtt{0}w)_7 \mid k \geqslant 0,\ x_i \in \{\mathtt{3},\mathtt{6}\}^*,\ \exists e \in \mathbb{N}\ Resolve(x_1,\dots,x_k) = \tau_2(e),\ (w)_7 \in C_{\tau_2(e)}\}\\ &= \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\dots \mathtt{1}x_1\mathtt{1}\mathtt{0}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*\ (w)_7 \in B_{f_{\tau_2^{-1}(Resolve(x_1,\dots,x_k))}((x_{k+1})_2)}\}\\ &= \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\dots \mathtt{1}x_1\mathtt{1}\mathtt{0}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*\ (w)_7 \in B_{Resolve(x_1,\dots,x_k,x_{k+1})}\}\end{aligned}$$

$$\begin{aligned}&Append_{\mathtt{3},\mathtt{6}}(Y)\\ &= Append_{\mathtt{3},\mathtt{6}}(\{(x\mathtt{1}x_k\mathtt{1}x_{k-1}\dots \mathtt{1}x_1\mathtt{1}\mathtt{0}w)_7 \mid k \geqslant 0, x, x_i \in \{\mathtt{3},\mathtt{6}\}^*,\ \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*(w)_7 \in B_{Resolve(x_1,\dots,x_{k+1})}\})\\ &= \{(x\mathtt{1}x_k\mathtt{1}x_{k-1}\dots \mathtt{1}x_1\mathtt{1}\mathtt{0}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, x \in \{\mathtt{3},\mathtt{6}\}^+,\ \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*(w)_7 \in B_{Resolve(x_1,\dots,x_k,x_{k+1})}\}\end{aligned}$$

And taking the union of those two we obtain

$$\begin{aligned}&\{(\mathtt{1}x_k\mathtt{1}x_{k-1}\dots \mathtt{1}x_1\mathtt{1}\mathtt{0}w)_7 \mid k \geqslant 0,\ x_i \in \{\mathtt{3},\mathtt{6}\}^*, \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*\ (w)_7 \in B_{Resolve(x_1,\dots,x_k,x_{k+1})}\}\\ &\cup \{(x\mathtt{1}x_k\mathtt{1}x_{k-1}\dots \mathtt{1}x_1\mathtt{1}\mathtt{0}w)_7 \mid k \geqslant 0,\ x_i \in \{\mathtt{3},\mathtt{6}\}^*,\ x \in \{\mathtt{3},\mathtt{6}\}^+,\ \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*\ (w)_7 \in B_{Resolve(x_1,\dots,x_k,x_{k+1})}\}\\ &= \{(x\mathtt{1}x_k\mathtt{1}x_{k-1}\dots \mathtt{1}x_1\mathtt{1}\mathtt{0}w)_7 \mid k \geqslant 0,\ x, x_i \in \{\mathtt{3},\mathtt{6}\}^*,\ \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^*\ (w)_7 \in B_{Resolve(x_1,\dots,x_k,x_{k+1})}\}\end{aligned}$$

And so we have shown (4.4).

Consider the first part of (4.5) first. Then

$$\begin{aligned}Remove_1(Goal_0 \cap Admissible) &= Remove_1(Goal_0)\\ &= \{(x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 1, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in B_{Resolve(x_1,\dots,x_k)}\}\\ &= \{(x_{k+1} \mathtt{1} x_k \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in B_{Resolve(x_1,\dots,x_{k+1})}\}\end{aligned}$$

and the left part of it is

$$\begin{aligned}Y &= \{(x \mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0, x, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* (w)_7 \in B_{Resolve(x_1,\dots,x_{k+1})}\}\\ &\subseteq \{(x \mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0, x, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in B_{Resolve(x_1,\dots,x_k,x)}\}\\ &= \{(x_{k+1} \mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in B_{Resolve(x_1,\dots,x_{k+1})}\}\\ &= Remove_1(Goal_0)\end{aligned}$$

And the second inclusion

$$\begin{aligned}&Y \cup \widetilde{Y} =\\ &\{(x \mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0, x, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* (w)_7 \in B_{Resolve(x_1,\dots,x_k,x_{k+1})}\}\\ &\cup \{(x \mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0, x, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* (w)_7 \in C_{Resolve(x_1,\dots,x_k,x_{k+1})}\}\\ &= \{(x \mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0, x, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* (w)_7 \notin C_{Resolve(x_1,\dots,x_{k+1})}\}\\ &\cup \{(x \mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0, x, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* (w)_7 \in C_{Resolve(x_1,\dots,x_{k+1})}\}\\ &= \{(x \mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0, x, x_i \in \{\mathtt{3},\mathtt{6}\}^*,\\ &\quad \exists x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* (w)_7 \in C_{Resolve(x_1,\dots,x_{k+1})} \text{ or}\\ &\quad \forall x_{k+1} \in \{\mathtt{3},\mathtt{6}\}^* (w)_7 \notin C_{Resolve(x_1,\dots,x_{k+1})}\}\\ &= \{(x \mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0, x, x_i \in \{\mathtt{3},\mathtt{6}\}^*, Resolve(x_1, \dots x_{k+1}) \text{is defined}\}\\ &\supseteq \{(x_{k+1} \mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*,\ (w)_7 \in B_{Resolve(x_1,\dots x_k,x_{k+1})}\}\\ &= Remove_1(Goal_0)\end{aligned}$$

And thus both parts of (4.5) were checked.

(4.6) and (4.7) hold trivially.

For (4.8) note that

$$\begin{aligned}&Goal_1 \cap R_0\\ &= \{(\mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in C_{Resolve(x_1,\dots,x_k)}\}\\ &\cap \{(\mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0,\ x_i \in \{\mathtt{3},\mathtt{6}\}^*, \exists e \in \mathbb{N}, Resolve(x_1, \dots, x_k) = \tau_1(e) (w)_7 \in B_{\tau_1(e)}\}\\ &= \{(\mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \exists e \in \mathbb{N}, Resolve(x_1, \dots, x_k) = \tau_1(e) (w)_7 \in C_{\tau_1(e)} \cap B_{\tau_1(e)}\}\\ &= \varnothing\end{aligned}$$

Similarly for (4.9)

$$\begin{aligned}&Goal_0 \cap R_1\\ &= \{(\mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in B_{Resolve(x_1,\dots,x_k)}\}\\ &\cap \{(\mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0,\ x_i \in \{\mathtt{3},\mathtt{6}\}^*, \exists e \in \mathbb{N}, Resolve(x_1, \dots, x_k) = \tau_1(e) (w)_7 \in C_{\tau_1(e)}\}\\ &= \{(\mathtt{1} x_k \mathtt{1} x_{k-1} \dots \mathtt{1} x_1 \mathtt{10} w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, \exists e \in \mathbb{N}, Resolve(x_1, \dots, x_k) = \tau_1(e) (w)_7 \in B_{\tau_1(e)} \cap C_{\tau_1(e)}\}\\ &= \varnothing\end{aligned}$$

We now address the uniqueness of the solution. Consider some solution $(X_0, X_1, Y, \widetilde{Y}, Z)$. As $X_0, X_1 \subseteq \mathit{Admissible}$, these sets are uniquely defined by their intersection with sets of the form $(\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7$, where $\mathit{Resolve}(x_1, \dots, x_k) = e$ is defined. By Lemma 12

$$\begin{aligned} X_0 \cap (\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 &= \mathit{Goal}_0 \cap (\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 \\ X_1 \cap (\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 &= \mathit{Goal}_1 \cap (\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}\Sigma_7^*)_7 \end{aligned}$$

and thus $X_0 = \mathit{Goal}_0$ and $X_1 = \mathit{Goal}_1$.

Note, that by Lemma A (4.3)–(4.6) uniquely define $\widetilde{Y}$, Y and Z for fixed $X_0 \cap \mathit{Admissible}$ and X_1; we must check the assumption of the lemma for $X_0 \cap \mathit{Admissible}$ and X_1, though. Clearly $X_0 \cap \mathit{Admissible} \cap X_1 = \varnothing$. If $(x\mathtt{1}w)_7 \in \mathit{Remove}_\mathtt{1}(X_0 \cap \mathit{Admissible})$ and $(x'\mathtt{1}w)_7 \notin \mathit{Remove}_\mathtt{1}(X_0 \cap \mathit{Admissible})$ then $(x\mathtt{1}w)_7 = (x\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w')_7$ for some $x_1, \dots, x_k \in \{\mathtt{3},\mathtt{6}\}^*$ such that $\mathit{Resolve}(x_1, \dots, x_k, x)$ is defined.

Then $(x\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w')_7 \notin \mathit{Remove}_\mathtt{1}(X_0 \cap k\mathit{Admissible})$ means that

$$(w)_7 \notin B_{\mathit{Resolve}(x_1,\dots,x_k,x)} \iff (w)_7 \in C_{\mathit{Resolve}(x_1,\dots,x_k,x)},$$

which enforces

$$(x\mathtt{1}x_k\mathtt{1}\dots\mathtt{1}x_1\mathtt{10}w')_7 \in \mathit{Remove}_\mathtt{1}(X_1).$$

Thus the assumptions of Lemma A hold. This concludes the proof. ■

Lemma 13. *The value of the expression*

$$\begin{aligned} \mathit{Remove}_\mathtt{10}(Z) &= (Z \cap \{(\mathtt{10})_7\} + \{(\mathtt{10})_7\}) \\ &\cup \bigcup_{i \in \Sigma_7 \setminus \{\mathtt{0}\}} \bigcup_{t \in \{0,1,2\}} (Z \cap (\mathtt{10}i\Sigma_7^t(\Sigma_7^3)^*)_7) - (\mathtt{10}^*)_7 \cap (i\Sigma_7^t(\Sigma_7^3)^*)_7 \end{aligned}$$

on any $S \subseteq (\mathtt{10}(\Sigma_7^* \setminus \mathtt{0}\Sigma_7^*))_7$ *is* $\mathit{Remove}_\mathtt{1}(S) = \{(w)_7 \mid (\mathtt{10}w)_7 \in S\}$.

The expression $\mathit{Remove}_\mathtt{10}$ works similarly to $\mathit{Remove}_\mathtt{1}$, and the proof of this lemma is analogous to the proof of Lemma 3.

(p. 10) **Theorem 2.** *For every set $B \subseteq \mathbb{Z}$ in the hyper-arithmetical hierarchy there is a system of equations over subsets of $\mathbb{Z}$ using union, addition, singleton constants and the constants $\mathbb{N}$ and $-\mathbb{N}$, such that $(B, \dots)$ is its unique solution.*

Proof. We assume first that $B \subseteq \mathbb{N}$. Let $B = B_{i_0}$ according to enumeration of sets in hyper-arithmetical hierarchy. Construct a system of equation (4.1)–(4.9).

Then by Lemma 7 this system has a unique solution, and one of its component, for variable X_0 is set

$$\mathit{Goal}_0 = \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in B_{\mathit{Resolve}(x_1,\dots,x_k)}\}$$

Construct an additional equation

$$X = \mathit{Remove}_\mathtt{10}(X_0 \cap (\mathtt{10}\Sigma_7^*)_7)$$

As

$$\begin{aligned} \mathit{Goal}_0 \cap (\mathtt{10}\Sigma_7^*)_7 &= \{(\mathtt{1}x_k\mathtt{1}x_{k-1}\dots\mathtt{1}x_1\mathtt{10}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in B_{\mathit{Resolve}(x_1,\dots,x_k)}\} \cap (\mathtt{10}\Sigma_7^*)_7 \\ &= \{(\mathtt{10}w)_7 \mid k \geqslant 0, x_i \in \{\mathtt{3},\mathtt{6}\}^*, (w)_7 \in B_{\mathit{Resolve}(\langle\rangle)}\} \\ &= \{(\mathtt{10}w)_7 \mid (w)_7 \in B_{i_0}\} \end{aligned}$$

Hence, by Lemma 13, the unique solution of this equation is B_{i_0}.

Now, as in the proof of Theorem 1 using Lemma 1 enhanced by Corollary 1, allows representing B_{i_0} by a system of equations over sets of natural numbers, using union, addition and subtraction, with singleton constants.

For an arbitrary hyper-arithmetical set of integers, its positive and negative parts are first represented as shown above, and then Lemma 2, yields the system representing the actual set.

∎

Appendix D. Proofs from Section 5

Lemma 8. *A set $X \subseteq \mathbb{Z}$ satisfies an equation* (p. 10)

$$X + \{0,4,11\} = \bigcup_{\substack{i \in \{0,1,3,4,6,7,\\ 8,9,10,12,13\}}} \tau_i(\mathbb{Z}) \cup \{11\}$$

iff $X = \sigma(\widehat{X})$ for some $\widehat{X} \subseteq \mathbb{Z}$.

Proof. Denote

$$\textsc{track}_i(S) = \{n \mid 16n + i \in S\}.$$

A set S is said to have an *empty (full) track i* if $\textsc{track}_i(S) = \varnothing$ ($\textsc{track}_i(S) = \mathbb{Z}$, respectively).

⇚ Let X be any set that satisfies the equation. Then the sum $\textsc{track}_2(X + \{0,4,11\})$ has empty tracks 2, 5, 14 and 15:

$$\textsc{track}_2(X + \{0,4,11\}) = \textsc{track}_5(X + \{0,4,11\}) = \\ = \textsc{track}_{14}(X + \{0,4,11\}) = \textsc{track}_{15}(X + \{0,4,11\}) = \varnothing$$

For this condition to hold, X must have many empty tracks as well. To be precise, each track t with t, $t+4$ or $t+11 \pmod{16}$ being in $\{2,5,14,15\}$ must be an empty track in X. Calculating such set of tracks, $\{2,5,14,15\} - \{0,4,11\} \pmod{16} = \{1,2,3,4,5,7,10,11,14,15\}$ are the numbers of tracks that must be empty in X.

Similar considerations apply to track 11, as $\textsc{track}_{11}(X + \{0,4,11\}) = \{0\}$. For every track t with $t = 11$, $t+4 = 11 \pmod{16}$ or $t+11 = 11 \pmod{16}$, it must hold that the t-th track of X is either an empty track or $\textsc{track}_t(X) = \{0\}$. The latter must hold for at least one such t. Let us calculate all such tracks t: these are tracks with numbers $\{11\} - \{0,4,11\} \pmod{16} = \{0,7,11\}$. Since it tracks number 7 and 11 are already known to be empty, it follows that $\textsc{track}_0(X) = \{0\}$.

In order to prove that X is a valid encoding of some set, it remains to prove that tracks number $6,8,9,12$ in X are full. Consider first that $\textsc{track}_3(X + \{0,4,11\}) = \mathbb{Z}$. Let us calculate the track numbers t such that there is $t' \in \{0,4,11\}$ with $(t + t') \pmod{16} = 3$: these are $\{3\} - \{0,4,11\} \pmod{16} = \{3,8,15\}$. Since tracks $3,15$ are known to be empty, then

$$\mathbb{Z} = \textsc{track}_3(X + \{0,4,11\}) = \\ = \textsc{track}_3(X) \cup (\textsc{track}_{15}(X) + 1) \cup (\textsc{track}_8(X) + 1) = \\ = \varnothing \cup \varnothing \cup (\textsc{track}_8(X) + 1) = \textsc{track}_8(X) + 1,$$

and thus track 8 of X is full. The analogous argument is used to prove that tracks $12, 9, 6$ are full. Consider $\text{TRACK}_7(X + \{0, 4, 11\}) = \mathbb{Z}$. Then $\{7\} - \{0, 4, 11\} \pmod{16} = \{7, 3, 12\}$. Since it is already known that tracks $3, 7$ are empty, the track 12 is full:

$$\begin{aligned}\mathbb{Z} = \text{TRACK}_7(X + \{0, 4, 11\}) &= \\ &= \text{TRACK}_7(X) \cup \text{TRACK}_3(X) \cup (\text{TRACK}_{12}(X) + 1) = \\ &= \varnothing \cup \varnothing \cup (\text{TRACK}_{12}(X) + 1) = \text{TRACK}_{12}(X) + 1.\end{aligned}$$

In the same way consider $\text{TRACK}_9(X + \{0, 4, 11\}) = \mathbb{Z}$. Then $\{9\} - \{0, 4, 11\} \pmod{16} = \{9, 5, 14\}$ and tracks $5, 14$ are empty, thus track 9 is full:

$$\begin{aligned}\mathbb{Z} = \text{TRACK}_9(X + \{0, 4, 11\}) &= \\ &= \text{TRACK}_9(X) \cup \text{TRACK}_5(X) \cup (\text{TRACK}_{14}(X) + 1) = \\ &= \text{TRACK}_9(X) \cup \varnothing \cup \varnothing = \text{TRACK}_9(X).\end{aligned}$$

Now let us inspect $\text{TRACK}_{10}(X + \{0, 4, 11\})$. Then $\{10\} - \{0, 4, 11\} \pmod{16} = \{10, 6, 15\}$. Since the tracks $10, 15$ are empty, then the 6th track is full:

$$\begin{aligned}\mathbb{Z} = \text{TRACK}_{10}(X + \{0, 4, 11\}) &= \\ &= \text{TRACK}_{10}(X) \cup \text{TRACK}_6(X) \cup 1 + \text{TRACK}_{15}(X) = \\ &= \varnothing \cup \text{TRACK}_6(X) \cup \varnothing = \text{TRACK}_6(X).\end{aligned}$$

Thus it has been proved that $X = \sigma(\text{TRACK}_{13}(X))$.

⇚ It remains to show the converse, that is, that if $X = \sigma(\widehat{X})$, then

$$X + \{0, 4, 11\} = \bigcup_{\substack{i \in \{0,1,3,4,6,7,\\ 8,9,10,12,13\}}} \tau_i(\mathbb{Z}) \cup \{11\}.$$

Since $X = \bigcup_{i=0}^{15} \tau_i(\text{TRACK}_i(X))$, then

$$\begin{aligned}X + \{0, 4, 11\} = \Big(\bigcup_i \tau_i(\text{TRACK}_i(X)) + 0\Big) \cup \Big(\bigcup_i \tau_i(\text{TRACK}_i(X)) + 4\Big) \cup \\ \cup \Big(\bigcup_i \tau_i(\text{TRACK}_i(X)) + 11\Big),\end{aligned}$$

and Table 1 presents the form of each particular term in this union. Each ith row represents track number i in X, and each column labeled $+j$ for $j \in \{0, 4, 11\}$ corresponds to the addition of a number j. The cell (i, j) gives the set $\text{TRACK}_i(X) + j$ and the number of the track in which this set appears in the result (this is track $i + j \pmod{16}$). Then each ℓ-th track $X + \{0, 4, 11\}$ is obtained as a union of all the appropriate sets in the Table 1.

According to the table, the values of the set $\widehat{X}$ are reflected in three tracks of the sum $X + \{0, 4, 11\}$: in tracks 13, 1 and 8 (in the last two cases, with offset 1). However, at the same time the sum contains full tracks 1, 8 and 13, and the contributions of $\widehat{X}$ to the sum are subsumed by these numbers, as $\tau_{13}(\widehat{X}) \subseteq \tau_{13}(\mathbb{Z})$, $\tau_1(\widehat{X} + 1) \subseteq \tau_1(\mathbb{Z})$ and $\tau_8(\widehat{X} + 1) \subseteq \tau_8(\mathbb{Z})$. Therefore, the value of the expression does not depend on $\widehat{X}$. Taking the union of all entries

	+0	+4	+11
$0 : \{0\}$	$0 : \{0\}$	$4 : \{0\}$	$11 : \{0\}$
$6 : \mathbb{Z}$	$6 : \mathbb{Z}$	$10 : \mathbb{Z}$	$1 : \mathbb{Z}$
$8 : \mathbb{Z}$	$8 : \mathbb{Z}$	$12 : \mathbb{Z}$	$3 : \mathbb{Z}$
$9 : \mathbb{Z}$	$9 : \mathbb{Z}$	$13 : \mathbb{Z}$	$4 : \mathbb{Z}$
$12 : \mathbb{Z}$	$12 : \mathbb{Z}$	$0 : \mathbb{Z}$	$7 : \mathbb{Z}$
$13 : \widehat{X}$	$13 : \widehat{X}$	$1 : \widehat{X}+1$	$8 : \widehat{X}+1$

Table 1: Tracks in the sum $\sigma(\widehat{X}) + \{0, 4, 11\}$, only non-empty tracks of $\sigma(\widehat{X})$ are included.

of the Table 1 proves that $X + \{0, 4, 11\}$ equals

$$\bigcup_{\substack{i\in\{0,1,3,4,6,7,\\8,9,10,12,13\}}} \tau_i(\mathbb{Z}) \cup \{11\},$$

as stated in the lemma. ∎

Lemma 9. *For all sets $X, Y, Z \subseteq \mathbb{Z}$,* (p. 10)

$$\sigma(Y) + \sigma(Z) + \{0, 1\} = \sigma(X) + \sigma(\{0\}) + \{0, 1\} \textit{ iff } Y + Z = X$$

and

$$\sigma(Y) + \sigma(Z) + \{0, 2\} = \sigma(X) + \sigma(X) + \{0, 2\} \textit{ iff } Y \cup Z = X.$$

Proof. The goal is to show that that for all $Y, Z \subseteq \mathbb{Z}$, the sum

$$\sigma(Y) + \sigma(Z) + \{0, 1\}$$

encodes the set $Y + Z + 1$ on one of its tracks, while the contents of all other tracks does not depend on Y or on Z. Similarly, the sum

$$\sigma(Y) + \sigma(Z) + \{0, 2\}$$

has a track that encodes $Y \cup Z$, while the rest of its tracks also do not depend on Y and Z.

The common part of both of the above sums is $\sigma(Y) + \sigma(Z)$, so let us calculate it first. Since

$$\sigma(Y) = \{0\} \cup \tau_6(\mathbb{Z}) \cup \tau_8(\mathbb{Z}) \cup \tau_9(\mathbb{Z}) \cup \tau_{12}(\mathbb{Z}) \cup \tau_{13}(Y) \quad \text{and}$$
$$\sigma(Z) = \{0\} \cup \tau_6(\mathbb{Z}) \cup \tau_8(\mathbb{Z}) \cup \tau_9(\mathbb{Z}) \cup \tau_{12}(\mathbb{Z}) \cup \tau_{13}(Z),$$

by the distributivity of union, the sum $\sigma(Y)+\sigma(Z)$ is a union of 36 terms, each being a sum of two individual tracks. Every such sum is contained in a single track as well, and Table 3 gives a case inspection of the form of all these terms. Each of its six rows corresponds to one of the nonempty tracks of $\sigma(Y)$, while its six columns refer to the nonempty tracks in $\sigma(Z)$. Then the cell gives the sum of these tracks, in the form of the track number and track contents: that is, for row representing $\text{TRACK}_i(\sigma(Y))$ and for column representing $\text{TRACK}_j(\sigma(Z))$, the cell (i, j) represents the set $\text{TRACK}_i(\sigma(Y)) + \text{TRACK}_j(\sigma(Z))$, which is bound to be on track $i + j \pmod{16}$. For example, the sum of track 8 of $\sigma(Y)$ and track 9 of $\sigma(Z)$ falls onto track $1 = 8 + 9 \pmod{16}$ and equals

$$\tau_8(\mathbb{Z}) + \tau_9(\mathbb{Z}) = \{8 + 9 + 16(m + n) \mid m, n \in \mathbb{Z}\} = \{1 + 16n \mid n \in \mathbb{Z}\} = \tau_1(\mathbb{Z}),$$

	$\sigma(Y)$	$\sigma(Z)$	$\sigma(Y)+\sigma(Z)$	$\sigma(Y)+\sigma(Z)+\{0,1\}$	$\sigma(Y)+\sigma(Z)+\{0,2\}$
0	$\{0\}$	$\{0\}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1	$\varnothing$	$\varnothing$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
2	$\varnothing$	$\varnothing$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
3	$\varnothing$	$\varnothing$	?	$\mathbb{Z}$	$\mathbb{Z}$
4	$\varnothing$	$\varnothing$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
5	$\varnothing$	$\varnothing$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
6	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
7	$\varnothing$	$\varnothing$	$\varnothing$	$\mathbb{Z}$	$\mathbb{Z}$
8	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
9	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
10	$\varnothing$	$\varnothing$	$Y+Z+1$	$\mathbb{Z}$	$\mathbb{Z}$
11	$\varnothing$	$\varnothing$	$\varnothing$	$Y+Z+1$	$\mathbb{Z}$
12	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
13	Y	Z	$Y \cup Z$	$\mathbb{Z}$	$Y \cup Z$
14	$\varnothing$	$\varnothing$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
15	$\varnothing$	$\varnothing$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

Table 2: Tracks in the sums of $\sigma(Y)+\sigma(Z)$ with constants.

while adding track 13 of $\sigma(Y)$ to track 13 of $\sigma(Z)$ results in

$$\tau_{13}(Y)+\tau_{13}(Z)=\{26+16(m+n) \mid m \in Y,\ n \in Z\}=\tau_{10}(Y+Z+1),$$

which is reflected in the table. Each question mark denotes a track with unspecified contents. Though this contents can be calculated, it is actually irrelevant, because it does not influence the value of the subsequent sums $\sigma(Y)+\sigma(Z)+\{0,1\}$ and $\sigma(Y)+\sigma(Z)+\{0,2\}$.

Now the value of each i-th track of $\sigma(Y)+\sigma(Z)$ is obtained as the union of all sums in Table 3 that belong to the i-th track. The final values of these tracks are presented in the corresponding column of Table 2.

Now the contents of the tracks in $\sigma(Y)+\sigma(Z)+\{0,1\}$ can be completely described. The calculations are given in Table 2, and the result is that for all Y and Z,

$$\begin{aligned} \text{TRACK}_{11}(\sigma(Y)+\sigma(Z)+\{0,1\}) &= Y+Z+1 \\ \text{TRACK}_{i}(\sigma(Y)+\sigma(Z)+\{0,1\}) &= \mathbb{Z} \qquad \text{for } i \neq 11 \end{aligned}$$

It easily follows that

$$X = Y+Z$$

iff

$$\sigma(X)+\sigma(\{0\})+\{0,1\}=\sigma(Y)+\sigma(Z)+\{0,1\},$$

as, clearly, $X = X+\{0\}$.

For the set $\sigma(Y)+\sigma(Z)+\{0,2\}$, in the same way, for all Y and Z,

$$\begin{aligned} \text{TRACK}_{13}(\sigma(Y)+\sigma(Z)+\{0,2\}) &= Y \cup Z \\ \text{TRACK}_{j}(\sigma(Y)+\sigma(Z)+\{0,2\}) &= \mathbb{Z} \qquad \text{for } j \neq 13 \end{aligned}$$

and therefore for all X, Y, Z,

$$X = Y \cup Z$$

	$0:\{0\}$	$6:\mathbb{Z}$	$8:\mathbb{Z}$	$9:\mathbb{Z}$	$12:\mathbb{Z}$	$13:Z$
$0:\{0\}$	$0:\{0\}$	$6:\mathbb{Z}$	$8:\mathbb{Z}$	$9:\mathbb{Z}$	$12:\mathbb{Z}$	$13:Z$
$6:\mathbb{Z}$	$6:\mathbb{Z}$	$12:\mathbb{Z}$	$14:\mathbb{Z}$	$15:\mathbb{Z}$	$2:\mathbb{Z}$	$3:?$
$8:\mathbb{Z}$	$8:\mathbb{Z}$	$14:\mathbb{Z}$	$0:\mathbb{Z}$	$1:\mathbb{Z}$	$4:\mathbb{Z}$	$5:?$
$9:\mathbb{Z}$	$9:\mathbb{Z}$	$15:\mathbb{Z}$	$1:\mathbb{Z}$	$2:\mathbb{Z}$	$5:\mathbb{Z}$	$6:?$
$12:\mathbb{Z}$	$12:\mathbb{Z}$	$2:\mathbb{Z}$	$4:\mathbb{Z}$	$5:\mathbb{Z}$	$8:\mathbb{Z}$	$9:?$
$13:Y$	$13:Y$	$3:?$	$5:?$	$6:?$	$9:?$	$10:(Y+Z)+1$

Table 3: Tracks in the sum $\sigma(Y)+\sigma(Z)$. Question marks denote sets that depend on X or Y and whose actual values are unimportant.

iff

$$\sigma(X)+\sigma(X)+\{0,2\}=\sigma(Y)+\sigma(Z)+\{0,2\},$$

since $X = X \cup X$.

Both claims of the lemma follow. ■

Theorem 3. *For every hyper-arithmetical set $S \subseteq \mathbb{Z}$ there exists a system of equations* (p. 11)

$$\begin{cases} \varphi_1(X_1,\ldots,X_n) &= \psi_1(X_1,\ldots,X_n) \\ &\vdots \\ \varphi_m(X_1,\ldots,X_n) &= \psi_m(X_1,\ldots,X_n) \end{cases}$$

with φ_j, ψ_j using the operation of addition and ultimately periodic constants, which has a unique solution with $X_1 = T$, where $S = \{n \mid 16n \in T\}$.

Sketch of a proof. A system of equations with union and addition representing S exists by Theorem 2. This system is first decomposed to have all equations of the form $X = Y + Z$, $X = Y \cup Z$ or $X = C$. For every variable X of this system, the new system has a variable X' with an equation as in Lemma 8. Next, according to Lemma 9, the equations $Y + Z = X$, $Y \cup Z = X$ or $X = C$ are transformed to equations $Y' + Z' + \{0,1\} = X' + \sigma(\{0\}) + \{0,1\}$, $Y' + Z' + \{0,2\} = X' + X' + \{0,2\}$ and $X' = \sigma(C)$, respectively, and the resulting system should have a unique solution with $X' = \sigma(X)$. Thus the constructed system represents the set $\sigma(S)$, and adding an extra equation $X_1 = X + \{-13\}$ yields the set $T = \sigma(S) - 13$ with the desired properties. ■

Appendix E. Proofs from Section 6

Theorem 4. *The problem of whether a given system of equations over sets of integers with addition and ultimately periodic constants has a solution is Σ^1_1-complete.* (p. 11)

Proof. For any fixed system of equations, the statement that it has a solution naturally belongs to Σ^1_1: taking the arithmetical formula $Eq(X_1,\ldots,X_n)$, from Proposition 2, it suffices to write a second-order statement

$$(\exists X_1)\ldots(\exists X_n)\, Eq(X_1,\ldots,X_n).$$

Next, note that a given system can be effectively transformed to such a formula. Since the condition that a given Σ^1_1 closed formula is true can be specified by a certain universal

Σ^1_1 formula [21, Cor. 16-XX(a)], this leads to a Σ^1_1 formula representing the existence of solution of a system.

In order to prove that testing solution existence is Σ^1_1-hard, it is sufficient to reduce the problem from Proposition 6. Let M be the given Turing machine. Since $L(M) \in \Sigma^0_1$, there is a system of equations over sets of integers in variables $Y, Y_1, \ldots, Y_m$, which has a unique solution with $Y = L(M)$, and this system can be effectively constructed from the description of M. Introducing extra variables X and Y, consider the following additional equations, where the expressions $\mathit{Append}_{\mathtt{3,6}}(Z)$ and $E(Z)$ are taken from Lemma 4 and Lemma E, respectively:

$$\begin{aligned} X &\subseteq Y \\ \{1\} &\subseteq X \\ Z &= X \cup \mathit{Append}_{\mathtt{3,6}}(Z) \\ X &= E(Z) \end{aligned}$$

The variable X represents a subset of Y containing exactly the set of finite prefixes of a certain infinite sequence. It is then asserted that the number 1 corresponding to the empty prefix is in X, that the set Z is defined by $\{(x\mathtt{1}w)_\mathtt{7} \mid x \in \{\mathtt{3},\mathtt{6}\}^*,\ (\mathtt{1}w)_\mathtt{7} \in X\}$, and that $E(Z) = \{(\mathtt{1}w)_\mathtt{7} \mid \exists x \in \{\mathtt{3},\mathtt{6}\}^*,\ (x\mathtt{1}w)_\mathtt{7} \in X,\ (\mathtt{1}w)_\mathtt{7} \in X\}$ is exactly X, that is, that every element of X can be extended to an element of X. ∎

(p. 11) **Proposition 7.** *The problem of whether a given system of equations over sets of integers using addition and ultimately periodic constants has a unique solution can be represented as a conjunction of a Σ^1_1-formula and a Π^1_1-formula, and is accordingly in Δ^1_2.*

Proof. The property of having at most one solution can be expressed by the following Π^1_1-formula:

$$(\forall X_1)\ldots(\forall X_n)(\forall X'_1)\ldots(\forall X'_n)\big[Eq(X_1,\ldots,X_n) \wedge Eq(X'_1,\ldots,X'_n)\big] \to (\forall n)(\forall i) n \in X_i \leftrightarrow n \in X'_i$$

Then the condition of having a unique solution is a conjunction of the latter formula with the Σ^1_1-formula expressing solution existence. The resulting conjunction can be reformulated both as a Σ^1_2 and as a Π^1_2 formula, which proves the theorem. ∎